A LINDEMANN-WEIERSTRASS THEOREM FOR E-FUNCTIONS

by

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Abstract. — E-functions were introduced by Siegel in 1929 to generalize Diophantine properties of the exponential function. After developments of Siegel's methods by Shidlovskii, Nesterenko and André, Beukers proved in 2006 an optimal result on the algebraic independence of the values of E-functions. Since then, it seems that no general result was stated concerning the relations between the values of a single E-function. We prove that André's theory of E-operators and Beuker's result lead to a Lindemann–Weierstrass theorem for E-functions. As a consequence, we show that all transcendental values at algebraic arguments of an entire hypergeometric function are linearly independent over $\overline{\mathbb{Q}}$.

1. Introduction

Hermite proved in 1873 that e is a transcendental number. Nine years later, Lindemann generalized this result and obtained a statement that contains the transcendence of both numbers e and π: if α is a non-zero algebraic number then e^α is transcendental. In his original memoir of 1882, Lindemann sketched a more general result which was rigorously proved by Weierstrass and published in 1885.

***Theorem A* (Lindemann–Weierstrass).** — *If $\alpha_1, \dots, \alpha_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

This theorem has the following equivalent formulation [**5**], which is the main motivation of this article.

***Theorem B* (Lindemann–Weierstrass).** — *If $\alpha_1, \dots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

In order to generalize the Lindemann–Weierstrass theorem to other special functions, Siegel introduced in 1929 [**18**] the notion of *E-function*. He proved in particular a Hermite–Lindemann theorem for the Bessel function J_0: if α is a non-zero algebraic number then $J_0(\alpha)$ is transcendental. Siegel also obtained a Lindemann–Weierstrass analog:

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If $\alpha_1^2, \dots, \alpha_n^2$ are pairwise distinct non-zero algebraic numbers, then $J_0(\alpha_1), \dots, J_0(\alpha_n)$ are algebraically independent over $\mathbb{Q}$.

The additional condition $\alpha_i \neq \pm\alpha_j$ is natural for the Bessel function since $J_0(-z) = J_0(z)$.

Definition 1.1. — *An E-function* (*) *is an entire function $f(z)$ that satisfies a non-trivial linear differential equation with coefficients in $\overline{\mathbb{Q}}[z]$ and which has a power series expansion of the form*

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

whose coefficients a_n are algebraic numbers and satisfy the following growth condition: letting $d_n \geq 1$ denote, for every $n \geq 0$, the smallest integer such that $d_n a_0, \dots, d_n a_n$ are algebraic integers, there exists a real number $C > 0$ such that, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all $n \geq 1$, we have $|\sigma(a_n)| \leq C^n$ and $d_n \leq C^n$.

Classical examples of E-functions are the exponential, Bessel functions and more generally hypergeometric functions (*e.g.* [**20**]) (†)

$$F_{\mathbf{a},\mathbf{b}}(z^{s-r}) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} z^{(s-r)n}, \tag{1.1}$$

with integers $s > r \geq 0$, $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Q}^r$ and $\mathbf{b} = (b_1, \dots, b_s) \in (\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^s$. The set of E-functions forms a ring and a $\overline{\mathbb{Q}}[z]$-module stable by transformations $f(z) \mapsto f(\lambda z)$ with $\lambda \in \overline{\mathbb{Q}}$. It is also useful to have in mind that an E-function is either polynomial or transcendental over $\mathbb{C}(z)$.

The methods of Siegel were successively improved by Shidlovskii [**19**], Nesterenko–Shidlovskii [**14**], André [**3**] and Beukers [**6**] to finally reach in 2006 the following deep result.

***Theorem C* (Beukers [6])**. — *Let $Y = (f_1, \dots, f_n)^\top$ be a vector of E-functions satisfying a system of first order equations $Y' = AY$ where A is an $n \times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote the common denominator of the entries of A by $T(z)$. Let α be an algebraic number satisfying $\alpha T(\alpha) \neq 0$. Then, for any homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ such that $P(f_1(\alpha), \dots, f_n(\alpha)) = 0$, there exists a polynomial $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the variables $X_1, \dots, X_n$, such that $Q(\alpha, X_1, \dots, X_n) = P(X_1, \dots, X_n)$ and $Q(z, f_1(z), \dots, f_n(z)) = 0$.*

Since then, it seems that no general result was stated concerning the relations between the values of a single E-function. In this article, we use Beukers' lifting result and André's theory of E-operators to prove a Lindemann–Weierstrass theorem for E-functions. As a consequence, we obtain the following generalization of Theorem A.

Theorem 1.2. — *Let $F_{\mathbf{a},\mathbf{b}}$ be an entire hypergeometric function, with $\mathbf{a} \in \mathbb{Q}^r$, $\mathbf{b} \in (\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^s$ where r and s are integers satisfying $s > r \geq 0$. Let $\alpha_1, \dots, \alpha_n$ be pairwise distinct algebraic numbers such that, for all i, $1 \leq i \leq n$, $F_{\mathbf{a},\mathbf{b}}(\alpha_i)$ is transcendental. Then the numbers $1, F_{\mathbf{a},\mathbf{b}}(\alpha_1), \dots, F_{\mathbf{a},\mathbf{b}}(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

(*) E-functions have been first defined by Siegel in 1929 in a more general way. André [**4**] generalized Theorem C to E-functions in Siegel's sense, see also [**13**].

(†) Galochkin gave in [**9**] a necessary and sufficient condition for a generalized hypergeometric function to be an E-function, which essentially reduces to the rationality of the parameters a_i and b_j.

Theorem 1.2 is a corollary of Theorem 4.2 stated in Section 4. Observe that the exponential function corresponds to the parameters $r = 0$, $s = 1$ and $\mathbf{b} = (1)$.

2. Diophantine properties of the values of E-functions

2.1. Hermite–Lindemann and E-functions. — In contrast to exp and J_0, a transcendental E-function f can take algebraic values at non-zero algebraic arguments. We call *exceptional values* those non-zero algebraic numbers α such that $f(\alpha) \in \overline{\mathbb{Q}}$. For example [**7**], for every $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and every positive integer d, the transcendental hypergeometric E-function

$$\sum_{n=0}^{\infty} \binom{n+d}{d} \frac{1}{(a+d+1)_n} z^n$$

has d exceptional values. The set of exceptional values of an entire function f is denoted by $\mathrm{Exc}(f)$. We say that f is *purely transcendental* if it has no exceptional value.

Since f satisfies a non-trivial linear differential equation over $\overline{\mathbb{Q}}[z]$, there exists a number field $\mathbb{K}$ which contains all its Taylor coefficients. Based on Beukers' lifting result, one can show [**8**] that for any exceptional value α, $f(\alpha) \in \mathbb{K}(\alpha)$. More precisely, Bostan, Rivoal and Salvy proved (see [**15**] and [**7**]) the following *canonical decomposition*: every transcendental E-function can be written in a unique way as $f = p + qg$ with $p, q \in \overline{\mathbb{Q}}[z]$, q monic and $q(0) \neq 0$, $\deg(p) < \deg(q)$ and g is a purely transcendental E-function. In particular, the exceptional values of f are the roots of q. A Hermite–Lindemann theorem for f thus comes down to finding its canonical decomposition or at least the roots of q.

By applying Beukers' lifting result to the system associated with the minimal inhomogeneous differential equation for f, Adamczewski and Rivoal [**1**] proved that there exists an algorithm to perform the following tasks. Given an E-function f as input [(‡)], it first says whether f is transcendental or not. If it is transcendental, it then outputs $\mathrm{Exc}(f)$. This algorithm was recently improved in [**7**]. Together with Rivoal's notes [**15**], this yields an algorithm to compute the canonical decomposition of any given E-function.

2.2. A Lindemann–Weierstrass theorem for E-functions. — As explained above, the first obstruction to generalize Theorem B is that at most one of the algebraic numbers $\alpha_1, \ldots, \alpha_n$ can be an exceptional value for f. The second obstruction is that an E-function may satisfy difference equations such as $J_0(-x) = J_0(x)$ or $\sin(-x) = -\sin(x)$, so we must avoid the relations $\alpha_i \neq \pm\alpha_j$ in those cases. More generally, given any E-function f and any non-zero distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$, we can construct an E-function

$$g(z) = L_1(z) f\left(\frac{z}{\alpha_1}\right) + \cdots + L_n(z) f\left(\frac{z}{\alpha_n}\right),$$

where L_i is the Lagrange polynomial of degree n such that $L_i(\alpha_j)$ is the Kronecker delta δ_{ij}. It follows that $g(\alpha_1) = \cdots = g(\alpha_n) = f(1)$. As proven below, this situation is in fact encoded by the singularities of the underlying G-functions or, equivalently, by the asymptotics of f at infinity (see [**8**]).

[(‡)] The input of the algorithm is a differential equation together with sufficiently initial conditions to define the E-function f. But an oracle has to guarantee that f is indeed an E-function.

To every E-function f corresponds a G-series $\psi(f)$ defined formally on the Taylor expansions at 0 [(§)] by

$$\psi\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n\right)=\sum_{n=0}^{\infty}a_nz^n.$$

This series has a positive radius of convergence and satisfies a non-trivial linear differential equation over $\overline{\mathbb{Q}}[z]$. It follows that $\psi(f)$ has finitely many singularities at finite distance, the set of which we denote by $\mathfrak{S}(f)$. Our main result is the following.

Theorem 2.1. — *Let $f_1,\dots,f_n$ be E-functions with pairwise disjoint sets $\mathfrak{S}(f_i)$. Let α be an algebraic number which is not an exceptional value of any of the f_i's. Then the numbers $1,f_1(\alpha),\dots,f_n(\alpha)$ are linearly independent over $\overline{\mathbb{Q}}$.*

The conclusion of Theorem 2.1 can be rephrased as follows: any non-trivial $\overline{\mathbb{Q}}$-linear combination of $f_1(\alpha),\dots,f_n(\alpha)$ is a transcendental number.

Under the assumtions of Theorem 2.1, it is not hard to prove that the E-functions $f_1,\dots,f_n$ are $\overline{\mathbb{Q}}(z)$-linearly independent unless two of them are polynomial. However, to naively prove the $\overline{\mathbb{Q}}$-linear independence of the values $f_1(\alpha),\dots,f_n(\alpha)$ with Beukers' lifting result, one has to construct an appropriate differential system which possibly brings new functions or has non-zero singularities. Somehow, Theorem 2.1 shows that those difficulties can essentially be overcome.

As a corollary, we obtain the following result on the linear independence of the values of E-functions evaluated at different algebraic arguments.

Corollary 2.2. — *Let $f_1,\dots,f_n$ be E-functions. Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers such that, for all $i\neq j$, $f_i(\alpha_i)$ is transcendental and $\alpha_i/\alpha_j\neq\rho_1/\rho_2$ for all $\rho_1\in\mathfrak{S}(f_i)$, $\rho_2\in\mathfrak{S}(f_j)$. Then the numbers $1,f_1(\alpha_1),\dots,f_n(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof of Corollary 2.2. — We set $g_i(z):=f_i(\alpha_iz)$ for all i. We also have $\psi(g_i)(z)=\psi(f_i)(\alpha_iz)$ so $\mathfrak{S}(g_i)$ is the set of the numbers ρ/α_i for ρ in $\mathfrak{S}(f_i)$. If $\rho_1\in\mathfrak{S}(f_i)$ and $\rho_2\in\mathfrak{S}(f_j)$, then $\rho_1/\alpha_i\neq\rho_2/\alpha_j$ by hypotheses. In particular the sets $\mathfrak{S}(g_1),\dots,\mathfrak{S}(g_n)$ are pairwise disjoint and we can apply Theorem 2.1 to $g_1,\dots,g_n$ with $\alpha=1$. □

We obtain a Lindemann–Weierstrass theorem for E-functions by setting $f_1=\cdots=f_n$ in Corollary 2.2.

Corollary 2.3. — *Let f be an E-function. Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers such that, for all $i\neq j$, $f(\alpha_i)$ is transcendental and $\alpha_i/\alpha_j\neq\rho_1/\rho_2$ for all $\rho_1,\rho_2\in\mathfrak{S}(f)$. Then the numbers $1,f(\alpha_1),\dots,f(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

In the case of the exponential function, which is purely transcendental, we have $\psi(\exp)=1/(1-z)$ so $\mathfrak{S}(\exp)=\{1\}$. The non-nullity assumption on the α_i's in Corollary 2.3 is counterbalanced by the presence of 1 in the conclusion. We retrieve Theorem B. The Bessel function J_0 is also purely transcendental and we have $\psi(J_0)=1/\sqrt{1+z^2}$ so $\mathfrak{S}(J_0)=\{-i,i\}$. We retrieve the $\overline{\mathbb{Q}}$-linear independence of $J_0(\alpha_1),\dots,J_0(\alpha_n)$ when $\alpha_i\neq\pm\alpha_j$. We describe in Section 4 several applications of Theorem 2.1 and its corollaries, in particular we obtain that

[(§)] As pointed out by André in [**3**], the operator ψ can be obtained *via* the Laplace transform. If f is an E-function, then there exists a positive constant ρ such that $|f(z)|=O(e^{\rho|z|})$. Hence the Laplace transform $f^+(z)=\int_0^\infty f(w)e^{-wz}\mathrm{d}w$ is well-defined for $\Re(z)>\rho$. A standard calculation shows that $\psi(f)(z)=\frac{1}{z}f^+(\frac{1}{z})$.

the transcendental values at algebraic arguments of an entire hypergeometric E-function are linearly independent over $\overline{\mathbb{Q}}$.

We prove in Section 4.2 another straightforward consequence of Corollary 2.2: given transcendental E-functions $f_1, \dots, f_n$, and non-zero algebraic functions $\mu_1, \dots, \mu_n, \eta_1, \dots, \eta_n$ over $\overline{\mathbb{Q}}(z)$ such that the η_i's are non-constant and pairwise linearly independent over $\overline{\mathbb{Q}}$, then the function

$$\mu_1 \cdot (f_1 \circ \eta_1) + \cdots + \mu_n \cdot (f_n \circ \eta_n)$$

is transcendental over $\overline{\mathbb{Q}}(z)$. This is a broad generalization of the fact that E-functions cannot be solutions of non-trivial linear difference equations over $\overline{\mathbb{Q}}(z)$ associated with a Mahler or a shift operator (*e.g.* [**2**]).

3. Proof of Theorem 2.1

Let $f_1, \dots, f_n$ be E-functions such that, for all distinct i and j, the sets $\mathfrak{S}(f_i)$ and $\mathfrak{S}(f_j)$ are disjoint. Let α be a fixed non-zero algebraic number such that $f_1(\alpha), \dots, f_n(\alpha)$ are transcendental numbers. Let us assume that there is a $\overline{\mathbb{Q}}$-linear relation between the numbers $1, f_1(\alpha), \dots, f_n(\alpha)$ given by

$$\lambda_0 + \lambda_1 f_1(\alpha) + \cdots + \lambda_n f_n(\alpha) = 0.$$

We shall show that this relation is trivial, that is $\lambda_0 = \cdots = \lambda_n = 0$.

We write f_0 for the constant E-function equal to 1. Let i be fixed in $\{0, \dots, n\}$. By the results of André on E-operators [**3**], there exists a differential operator L_i annihilating f_i and with only 0 and ∞ as singularities. We write s_i for the differential order of L_i and $Y_i = (f_i, \dots, f_i^{(s_i-1)})^\top$. It follows that Y_i is solution of a differential system $Y' = A_i Y$ where A_i is a matrix with entries in $\overline{\mathbb{Q}}[z, 1/z]$. The direct sum of those systems gives a system $Y' = AY$, where the entries of A belong to $\overline{\mathbb{Q}}[z, 1/z]$, and with the concatenation of the Y_i's as a vector solution of E-functions. Since α is non-zero, it is not a singularity of A and we can apply Theorem C: for every i, there exists a linear differential operator $\mathcal{L}_i$ with coefficients in $\overline{\mathbb{Q}}[z]$ such that

$$\mathcal{L}_0 f_0 + \cdots + \mathcal{L}_n f_n = 0,$$

and with $(\mathcal{L}_i f_i)(\alpha) = \lambda_i f_i(\alpha)$ for all i. Since ψ is clearly additive, we also have

$$\psi(\mathcal{L}_0 f_0) + \cdots + \psi(\mathcal{L}_n f_n) = 0.$$

Let us fix i in $\{0, \dots, n\}$. We claim that the singularities of $\psi(\mathcal{L}_i f_i)$ are singularities of $\psi(f_i)$. Indeed, by induction on the differential order and the degree in z of $\mathcal{L}_i$, it suffices to prove the claim for the differential operators defined by $\beta \in \overline{\mathbb{Q}}$, z and $\frac{d}{dz}$. Calculations give the useful formulas (*e.g.* [**3**, Section 2])

$$\psi(z f_i(z)) = \left(z^2 \frac{d}{dz} + z\right) \psi(f_i) \quad \text{and} \quad \psi\left(\frac{d}{dz} f_i(z)\right) = \frac{\psi(f_i)(z) - f_i(0)}{z}.$$

In all cases, we see that the singularities at finite distance of $\psi(\mathcal{L} f_i)$ are singularities of $\psi(f_i)$, so $\mathfrak{S}(\mathcal{L}_i f_i) \subseteq \mathfrak{S}(f_i)$. It follows that for all distinct i and j, the sets $\mathfrak{S}(\mathcal{L}_i f_i)$ and $\mathfrak{S}(\mathcal{L}_j f_j)$ are disjoint.

Let i be fixed in $\{0,\dots,n\}$. we obtain that

$$\psi(\mathcal{L}_i f_i) = -\sum_{\substack{j=1\\ j\neq i}}^{n} \psi(\mathcal{L}_j f_j)$$

is a G-function with no singularity at finite distance. Due to the growth conditions on a_n in Definition 1.1, a G-function with no singularity at finite distance is a polynomial in $\overline{\mathbb{Q}}[z]$. Hence $\psi(\mathcal{L}_i f_i) \in \overline{\mathbb{Q}}[z]$ and we also have $\mathcal{L}_i f_i \in \overline{\mathbb{Q}}[z]$. As a consequence, for all $i \in \{0,\dots,n\}$, we obtain that

$$\lambda_i f_i(\alpha) = (\mathcal{L}_i f_i)(\alpha) \in \overline{\mathbb{Q}}.$$

For all i in $\{1,\dots,n\}$, $f_i(\alpha)$ is transcendental so $\lambda_i = 0$. It follows that $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$ and the relation is trivial. □

4. Applications of Theorem 2.1

4.1. Hypergeometric E-functions. — We recall that hypergeometric E-functions $F_{\mathbf{a},\mathbf{b}}(z^{s-r})$ are defined by (1.1) and that $(a)_n$ denotes the Pocchammer symbol defined by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for positive integers n. Since the only singularity at finite distance of a hypergeometric G-function is 1, we obtain the following Diophantine results for their values.

Theorem 4.1. — *For all $i \in \{1,\dots,n\}$, let $s_i > r_i \geq 0$ be integers, set $k_i = s_i - r_i$ and consider parameters $\mathbf{a}_i \in \mathbb{Q}^{r_i}$ and $\mathbf{b}_i \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^{s_i}$. Let $\alpha_1,\dots,\alpha_n$ be non-zero algebraic numbers such that, for all $i \neq j$, $F_{\mathbf{a}_i,\mathbf{b}_i}(\alpha_i)$ is transcendental and $\alpha_i^{k_j}/\alpha_j^{k_i} \neq (k_j/k_i)^{k_i k_j}$. Then the numbers $1, F_{\mathbf{a}_1,\mathbf{b}_1}(\alpha_1),\dots,F_{\mathbf{a}_n,\mathbf{b}_n}(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. — We consider $f_i(z) = F_{\mathbf{a}_i,\mathbf{b}_i}(z^{k_i})$ which is a hypergeometric E-function. The G-series $\psi(f_i)(z)$ is given by

$$\sum_{n=0}^{\infty}(k_i n)!\frac{(a_{i,1})_n\cdots(a_{i,r_i})_n}{(\beta_{i,1})_n\cdots(\beta_{i,s_i})_n}z^{k_i n} = \sum_{n=0}^{\infty} k_i^{k_i n}\left(\frac{1}{k_i}\right)_n\cdots\left(\frac{k_i}{k_i}\right)_n\cdot\frac{(a_{i,1})_n\cdots(a_{i,r_i})_n}{(\beta_{i,1})_n\cdots(\beta_{i,s_i})_n}z^{k_i n},$$

which is equal to $g_i((k_i z)^{k_i})$ where g_i is a hypergeometric G-function with only 1 as finite singularity. It follows that the singularities at finite distance of $\psi(f_i)$ are of the form ρ/k_i where ρ is a k_i-th root of unity. For all i, we write β_i for a k_i-th root of α_i. If $\rho_1 \in \mathfrak{S}(f_i)$ and $\rho_2 \in \mathfrak{S}(f_j)$ are such that $\beta_i/\beta_j = \rho_1/\rho_2$ then there are respective k_i-th and k_j-th roots of unity ξ_i and ξ_j such that $\beta_i/\beta_j = \xi_i k_j/(\xi_j k_i)$. By rising this equality to the power $k_i k_j$, we obtain that $\alpha_i^{k_j}/\alpha_j^{k_i} = (k_j/k_i)^{k_i k_j}$, which is a contradiction. Since $f_i(\beta_i) = F_{\mathbf{a}_i,\mathbf{b}_i}(\alpha_i)$ is transcendental, the result follows by Corollary 2.2. □

By focusing on parameters with same differences $s_i - r_i$, we obtain that the transcendental values of the associated hypergeometric functions are $\overline{\mathbb{Q}}$-linearly independent.

Theorem 4.2. — *Let k be a positive integer. For all $i \in \{1,\dots,n\}$, let $r_i \geq 0$ and $s_i = r_i + k$ be integers and consider parameters $\mathbf{a}_i \in \mathbb{Q}^{r_i}$ and $\mathbf{b}_i \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^{s_i}$. Let $\alpha_1,\dots,\alpha_n$ be pairwise distinct algebraic numbers such that, for all i, $1 \leq i \leq n$, $F_{\mathbf{a}_i,\mathbf{b}_i}(\alpha_i)$ is transcendental. Then the numbers $1, F_{\mathbf{a}_1,\mathbf{b}_1}(\alpha_1),\dots,F_{\mathbf{a}_n,\mathbf{b}_n}(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. — We can assume that the α_i's are non-zero. We set $f_i(z) := F_{\mathbf{a}_i,\mathbf{b}_i}(z^k)$ for all i. The singularities at finite distance of each $\psi(f_i)$ are of the form ρ/k where ρ is a k-th root of unity. Let $\beta_1, \dots, \beta_n$ be respective k-th roots of $\alpha_1, \dots, \alpha_n$. For all distinct i and j, the ratio β_i/β_j is not a k-th root of unity because $\alpha_i \neq \alpha_j$. The result follows by Corollary 2.2. □

The linear and algebraic independence of the values of hypergeometric E-functions and their derivatives have been extensively studied. General independence criteria have been obtained by Salikhov [**16, 17**], Galochkin [**10**] and recently Gorelov [**11, 12**]. Those results are stated with some restrictions on the parameters $\mathbf{a}_i$ and $\mathbf{b}_i$ or under the assumption that the α_i's are linearly independent over $\mathbb{Q}$. As far as we know, the absence of restrictions in our assumptions yields new results.

4.2. Algebraic pullbacks of E-functions. — We obtain the following functional result as a straightforward application of Corollary 2.2.

Proposition 4.3. — *Let $f_1, \dots, f_n$ be transcendental E-functions. Let $\mu_1, \dots, \mu_n, \eta_1, \dots, \eta_n$ be non-zero algebraic functions over $\overline{\mathbb{Q}}(z)$ such that the η_i's are non-constant and pairwise linearly independent over $\overline{\mathbb{Q}}$. Then the function*

$$\mu_1 \cdot (f_1 \circ \eta_1) + \cdots + \mu_n \cdot (f_n \circ \eta_n) \tag{4.1}$$

is transcendental over $\overline{\mathbb{Q}}(z)$.

Proof. — A non-zero algebraic function η has finitely many zeros. If in addition η is non-constant, then for each algebraic number α, the equation $\eta(z) = \alpha$ has only finitely many solutions. If η_i is well defined at $\alpha \in \overline{\mathbb{Q}}$, then $\eta_i(\alpha) \in \overline{\mathbb{Q}}$. Furthermore, f_i has only finitely many exceptional values. Thus there is a non-empty open subset $\mathcal{O}$ of $\mathbb{C}$ such that the μ_i's and η_i's are well defined and do not vanish on $\mathcal{O}$, and for every algebraic number $\alpha \in \mathcal{O}$ and every i, $f_i(\eta_i(\alpha))$ is a transcendental number. Assume that the function (4.1) is algebraic over $\overline{\mathbb{Q}}(z)$. Hence, for every algebraic numbers $\alpha \in \mathcal{O}$, we have

$$\mu_1(\alpha) f_1(\eta_1(\alpha)) + \cdots + \mu_n(\alpha) f_n(\eta_n(\alpha)) \in \overline{\mathbb{Q}}.$$

By Corollary 2.2, for each such α, there are $i \neq j$, $\rho_1 \in \mathfrak{S}(f_i)$ and $\rho_2 \in \mathfrak{S}(f_j)$ such that $\eta_i(\alpha)/\eta_j(\alpha) = \rho_1/\rho_2$. Since the f_i's are transcendental, the sets $\mathfrak{S}(f_i)$ are finite and non-empty. We obtain by the pigeonhole principle that there exist $i \neq j$, $\lambda \in \overline{\mathbb{Q}}$ and infinitely many α in $\mathcal{O}$ such that $\eta_i(\alpha)/\eta_j(\alpha) = \lambda$. This yields $\eta_i = \lambda\eta_j$ and ends the proof. □

4.3. A last illustration of Theorem 2.1. — Let us consider a last application of Corollary 2.2 to the sine integral.

Proposition 4.4. — *Let $\alpha_1^2, \dots, \alpha_n^2, \beta_1^2, \dots, \beta_n^2$ be pairwise distinct algebraic numbers. Then the numbers*

$$\int_{\alpha_i}^{\beta_i} \frac{\sin t}{t} dt, \quad 1 \leq i \leq n,$$

are transcendental and linearly independent over $\overline{\mathbb{Q}}$.

Proof. — Let us consider the sine integral and its associated G-series

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt \quad \text{and} \quad \psi(\mathrm{Si})(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1},$$

which is the Taylor expansion of arctan at the origin. It follows that $\mathfrak{S}(\mathrm{Si}) = \{-i, i\}$. Let j be fixed in $\{1, \dots, n\}$. By [**20**, Chapter 8], the sine integral is a purely transcendental E-function. Hence, if one of the α_j's or β_j's is zero, the corresponding integral is transcendental. Otherwise, we apply Corollary 2.3 to obtain that the numbers $1, \mathrm{Si}(\alpha_j)$ and $\mathrm{Si}(\beta_j)$ are $\overline{\mathbb{Q}}$-linearly independent. In particular, $\mathrm{Si}(\beta_j) - \mathrm{Si}(\alpha_j)$ is a transcendental number, as expected. Now it suffices to apply Corollary 2.3 to obtain that the numbers $\mathrm{Si}(\alpha_1), \dots, \mathrm{Si}(\alpha_n), \mathrm{Si}(\beta_1), \dots, \mathrm{Si}(\beta_n)$ are $\overline{\mathbb{Q}}$-linearly independent. □

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