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# REVERSE INEQUALITY FOR THE RIESZ TRANSFORMS ON RIEMANNIAN MANIFOLDS

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Abstract. Let $M$ be a complete Riemannian manifold satisfying the doubling volume condition for geodesic balls and $L^q$ scaled Poincaré inequalities on suitable remote balls for some $q < 2$. We prove the inequality $\left\|\Delta^{1/2} f\right\|_p \lesssim \|\nabla f\|_p$ for all $p \in (q, 2]$, which generalizes previous results due to Auscher and Coulhon. Our conclusion applies, in particular, when $M$ has a finite number of Euclidean ends. The proof strongly relies on Hardy inequalities, which are also new in this context and of independent interest.

## 1. Introduction

Throughout the paper, if $A(f)$ and $B(f)$ are two nonnegative quantities defined for all $f$ belonging to a set $E$, the notation $A(f) \lesssim B(f)$ means that there exists $C > 0$ such that $A(f) \leq CB(f)$ for all $f \in E$, while $A(f) \simeq B(f)$ means that $A(f) \lesssim B(f)$ and $B(f) \lesssim A(f)$.
Let $M$ be a complete connected noncompact Riemannian manifold. Denote by $\mu$ the Riemannian measure, by $\nabla$ the Riemannian gradient and by $\Delta$ the Laplace-Beltrami operator. The volume of a geodesic ball $B$ will be denoted by $V(B)$ instead of $\mu(B)$. In this work, we consider the following three inequalities for $p \in (1, \infty)$ (where the $L^p$-norms are computed with respect to the measure $\mu$):

*Date*: today.

$$||\Delta^{1/2}u||_p \lesssim ||\nabla u||_p \lesssim ||\Delta^{1/2}u||_p, \quad \forall u \in C_0^\infty(M) \tag{$\mathrm{E}_p$}$$

$$||\nabla u||_p \lesssim ||\Delta^{1/2}u||_p, \quad \forall u \in C_0^\infty(M) \tag{$\mathrm{R}_p$}$$

$$||\Delta^{1/2}u||_p \lesssim ||\nabla u||_p, \quad \forall u \in C_0^\infty(M) \tag{$\mathrm{RR}_p$}$$

It follows easily from the Green formula and the self-adjointness of $\Delta$ that

$$||\nabla u||_2^2 = (\Delta u, u) = ||\Delta^{1/2}||_2^2, \quad \forall u \in C_0^\infty(M).$$

Consequently, ($\mathrm{E}_p$) holds for $p = 2$ on any complete Riemannian manifold. The inequality ($\mathrm{R}_p$) is equivalent to the $L^p$-boundedness of the Riesz transform $\mathscr{R} = \nabla\Delta^{-1/2}$. A well-known duality argument, originally introduced in [3], shows that ($\mathrm{R}_p$) implies ($\mathrm{RR}_q$) for $q = p'$ the conjugate exponent, but the converse implication does not hold (see Section 1.1 below). The present work focuses on the inequality ($\mathrm{RR}_p$), which we establish for suitable ranges of $p$ in situations where inequalities of the form ($\mathrm{R}_{p'}$) do not hold. The proofs are strongly related to the geometry of the underlying manifold and to the behaviour of the heat kernel $p_t$, namely the kernel of the semigroup generated by $\Delta$.

We consider the case where $p_t$ satisfies Gaussian type pointwise upper estimates, and prove that, if a scaled $L^q$ Poincaré inequality holds on remote balls of $M$ for some $q \in [1, 2)$, then ($\mathrm{RR}_p$) for $p \in (q, 2]$ (see Section 1.1 below for precise statements).

1.1. **Previously known reverse Riesz inequalities.** In [2], P. Auscher and T. Coulhon have studied the inequality ($\mathrm{RR}_p$), and the relationship between ($\mathrm{R}_p$) and ($\mathrm{RR}_q$), $q = p'$. In order to recall some of their results, we need to introduce some geometric inequalities about $M$.
For all $x \in M$ and all $r > 0$, let $B(x, r)$ be the open geodesic ball with center $x$ and radius $r$ and set $V(x, r) := \mu(B(x, r))$. Say that the doubling volume property holds if and only if, for all $x \in M$ and all $r > 0$,

$$V(x, 2r) \lesssim V(x, r). \tag{D}$$

By iteration, this condition implies at once that there exists $D > 0$ such that for all $x \in M$ and all $0 < r < R$,

$$V(x, R) \lesssim \left(\frac{R}{r}\right)^D V(x, r). \tag{VD}$$

An easy consequence of (D) is that for every $0 < r \leq R$, and for every $x, y \in M$ such that $d(x, y) \leq r$, one has

$$V(x, R) \simeq V(y, R). \tag{1.1}$$

We also consider a reverse doubling volume condition: there exists $\nu > 0$ such that, for all $x \in M$ and all $0 < r < R$,

$$\left(\frac{R}{r}\right)^{\nu} V(x,r) \lesssim V(x,R). \tag{RD}$$

It is known that since $M$ is non-compact and connected, (D) implies (RD) for some $\nu > 0$.
Let $p \in [1,\infty)$. We consider the scaled $L^p$ Poincaré inequality on balls, namely :

$$||f - f_B||_{L^p(B)} \lesssim r||\nabla f||_{L^p(B)}, \quad f \in C^\infty(B), \forall B = B(x,r) \subset M, \tag{$P_p$}$$

where $f_B$ denotes the average of $f$ on $B$, that is $f_B := V(B)^{-1} \int_B f$.

Among other things, Auscher and Coulhon prove in [2] that

(i) if the Hodge projector onto exact 1-forms $\Pi := \mathscr{R}\mathscr{R}^*$ is $L^p$-bounded, then ($RR_{p'}$) implies ($R_p$),
(ii) if (D) holds, as well as ($P_q$) for some $q \in [1,2)$, then, ($RR_p$) holds for all $p \in (q,2)$.

As a consequence of (ii), one can see that the implication ($RR_{p'}$) $\Rightarrow$ ($R_p$) is false in general. Indeed, let $M$ be a complete Riemannian manifold of dimension $n \geq 3$, such that $M$ has only one end, and this end is asymptotically conic (see [12]); assume furthermore that the first eigenvalue of the cross-section of the corresponding cone is strictly less than $n-1$. Then, according to [12, Theorem 1.4], ($R_p$) holds on $M$ if and only if $p \in (1,p^*)$, where $n < p^* < +\infty$ only depends on the first eigenvalue of the cross-section of the cone. However, such a manifold $M$ satisfies ($P_1$), so that ($RR_p$) holds for all $p \in (1,2]$ according to (ii).
Observe that $(ii)$ above does not apply in the case where $M$ is, for instance, the connected sum of two copies of $\mathbb{R}^n$, since $(P_2)$ does not hold in this case ([10, Appendix]). As far as the Riesz transforms are concerned, it was shown in [5] that ($R_p$) holds on $M$ if and only if $1 < p < n$ if $n \geq 3$, and if and only if $1 < p \leq 2$ if $n = 2$, which implies that ($RR_p$) holds when $p > \frac{n}{n-1}$. However, the validity of ($RR_q$) for $1 < q \leq \frac{n}{n-1}$ remained open in that case.

1.2. **New results.** In the present work, we extend statement $(ii)$ above to the case where the $L^q$ Poincaré inequality only holds on some "remote" balls of $M$.
Let us fix once and for all a point $o$ in $M$. For $x \in M$, we will denote $r(x) = d(x,o)$. We let $B_0 = B(o,r_0)$, where $r_0 > 0$ is large enough and will be determined later.

**Definition 1.1.** *Let $x \in M$ and $r > 0$.*

(1) *The ball $B(x,r)$ is called* remote *if $r \leq \frac{r(x)}{2}$.*
(2) *The ball $B(x,r)$ is called* anchored *if $x = o$.*

(3) *The ball $B(x,r)$ is called* admissible *if either $B$ is remote, or $B(x,r)$ is anchored and $r \leq r_0$.*

In this article, instead of $L^p$ Poincaré inequalities ($\mathrm{P}_p$) for *all* balls of $M$, we will consider the following assumption that $L^p$ Poincaré inequalities hold only for certain balls:

**Definition 1.2.** We say that the $L^p$ Poincaré inequality holds *in the ends* of $M$ if, for every *admissible* balls $B$,

$$||f - f_B||_{L^p(B)} \lesssim r||\nabla f||_{L^p(B)}, \quad f \in C^\infty(B) \tag{$\mathrm{P}_p^E$}$$

where $r$ stands for the radius of $B$.

For $p = 2$, an assumption similar to ($\mathrm{P}_p^E$) has been considered in [14]. It follows from [13, Theorem 2.1] and the Hölder inequality that if $p \leq q$, ($\mathrm{P}_p^E$) $\Rightarrow$ ($\mathrm{P}_q^E$). See also the beginning of [13, Section 4]. Let us also point out that if the Ricci curvature has a quadratic lower bound of the form:

$$\mathrm{Ric}_x \geq -\frac{g}{1 + r(x)^2}, \tag{QD}$$

where $g$ is the Riemannian metric on $M$, then ($\mathrm{P}_p^E$) holds for all $p \geq 1$ (this follows from [19, Theorem 5.6.5]). In particular, ($\mathrm{P}_p^E$) holds for all $p \geq 1$ in the case where $M$ is the connected sum of two copies of $\mathbb{R}^n$.

Before stating our main theorem, we need to introduce the heat kernel $p_t(x,y)$, which is the kernel of the heat semigroup $e^{-t\Delta}$. Say that $p_t$ satisfies pointwise Gaussian upper bounds if

$$p_t(x,y) \lesssim \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall t > 0, \forall x, y \in M. \tag{UE}$$

It is well-known (see [7, Theorem 4]) that (UE) implies analogous estimates for the time-derivatives $\frac{\partial^n}{\partial t^n}$: for every $n \in \mathbb{N}$,

$$\left|\frac{\partial^n}{\partial t^n} p_t(x,y)\right| \lesssim \frac{1}{t^n V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall t > 0, \forall x, y \in M. \tag{1.2}$$

Sometimes we will use a slightly different (but equivalent, under (D)) version of (1.2), which we record here:

$$\left|\frac{\partial^n}{\partial t^n} p_t(x,y)\right| \lesssim \frac{1}{t^n V(y,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall t > 0, \forall x, y \in M. \tag{1.3}$$

(the constants that we call $c$ in (1.2) and (1.3) not necessarily being the same).

Say that $M$ has a finite number of ends if there exists an integer $N \geq 1$ such that, for all $R > 0$, $M \setminus B(o,R)$ has at most $N$ unbounded connected components. It is known ([4, Section[2.4.1]) that condition (D) implies that $M$ has a finite number of ends.

We also consider the following geometric condition.

**Definition 1.3.** We say that $(M, g)$ with a finite number of ends satisfies the Relative Connectedness in the Ends (RCE) condition, if there is a constant $\theta \in (0,1)$ such that for any point $x$ with $r(x) \geq 1$, there is a continuous path $c : [0,1] \to M$ satisfying

- $c(0) = x$.
- the length of $c$ is bounded by $\frac{r(x)}{\theta}$.
- $c([0,1]) \subset B(o, \theta^{-1} r(x)) \setminus B(o, \theta r(x))$.
- there is a geodesic ray $\gamma : [0, +\infty) \to M \setminus B(o, r(x))$ with $\gamma(0) = c(1)$.

When $M$ only has one end, the (RCE) condition is nothing but the (RCA) condition introduced in [11]. Let us also recall ([4, Theorem 2.4]) that, if (QD) and (RCE) hold, as well as the volume comparison property, namely

$$V(o,R) \lesssim V\left(x, \frac{R}{2}\right) \tag{VC}$$

for all $R \geq 1$ and all $x \in \partial B(o,R)$, then the relative Faber-Krahn inequality holds, hence (UE) and (D) hold.
The main purpose of this article is to show the following result:

**Theorem 1.4.** *Let $M$ be a complete Riemannian manifold satisfying (D), (UE), (RD) for some $\nu > 1$ and (RCE). Assume that for some $q \in (1,2]$ such that $q < \nu$, the $L^q$ Poincaré inequalities in the ends $(\mathrm{P}_q^E)$ hold. Then, for every $p \in [q,2)$, $(\mathrm{RR}_p)$ holds on $M$.*

**Question 1.5.** Is the assumption (RCE) in Theorem 1.4 really necessary?

**Remark 1.6.** Let us compare Theorem 1.4 with [2, Theorem 0.7]. Assume that (D), $(\mathrm{P}_q)$ for some $q \in [1,2]$ and (RD) for some $\nu > q$ hold. Then $(\mathrm{P}_2)$ holds as well; together with (D), it follows that (UE) holds (see [19, Theorem 4.2.6]). Moreover, [18, Proposition 0.3] shows that the conjunction of (D), $(\mathrm{P}_q)$ and (RD) for some $\nu > q$ imply the (RCA) condition. Since it is clear that $(\mathrm{P}_q) \Rightarrow (\mathrm{P}_q^E)$, it follows that for every $p \in (q,2)$, $(\mathrm{RR}_p)$ holds on $M$. In other words, under the condition (RD) for some $\nu > q$, assumptions in Theorem 1.4 are weaker than those of [2, Theorem 0.7].
Note also that, in Theorem 1.4, the conclusion $(\mathrm{RR}_p)$ holds for all $p \in [q,2)$, while the corresponding conclusion in [2, Theorem 0.7] under the assumption that $(\mathrm{P}_q)$ holds, is only stated for $p \in (q,2)$ (actually, a weak form of $(\mathrm{RR}_p)$ is proved for $p = q$ is proved in [2, Section 1.2]). However, when $(\mathrm{P}_q)$ holds, there exists $\varepsilon > 0$ such that $(\mathrm{P}_{q-\varepsilon})$ is also satisfied ([15, Theorem 1.0.1]), so that [2, Theorem 0.7] yields $(\mathrm{RR}_p)$ for $p = q$.

**Corollary 1.7.** *Let $M$ be a complete Riemannian manifold satisfying (QD), (VC), (RCE) and (RD) with $\nu > 1$. Then, for every $p \in (1, +\infty)$, $(\mathrm{RR}_p)$ holds on $M$.*

*Proof.* We are going to show that the assumptions of Theorem 1.4 are satisfied with $q = 1$. The $L^1$ Poincaré inequality in the ends follows from [19, Theorem 5.6.5]. Now, as we have mentioned before, (QD), (VC) and (RCE) imply (D) and (UE). Thus, the assumptions of Theorem 1.4 are satisfied, and therefore we get ($\mathrm{RR}_p$) for all $p \in (1, 2)$. The reverse inequalities ($\mathrm{RR}_p$) for $p \in [2, +\infty)$ follow from [6] and the implication ($\mathrm{R}_p$) $\Rightarrow$ ($\mathrm{RR}_q$), $q = p'$. □

The proof of Theorem 1.4 relies on three major ingredients: the first one (Proposition 3.1 below) is the covering of $M$ by admissible balls $(B_\alpha)_{\alpha\in\mathbb{N}}$ and the existence of an associated smooth partition of unity $(\chi_\alpha)_{\alpha\in\mathbb{N}}$. The second one (Theorem 2.3 below) is an $L^p$ Hardy inequality on $M$, obtained (roughly speaking) by "gluing" together local Poincaré inequalities thanks to a suitable covering. Our approach also uses a localized version of the Calderón-Zygmund decomposition in Sobolev spaces as in [2], already encountered in [9] (see Lemma 3.2 below).
The structure of the paper is as follows. Hardy inequalities are proved in Section 2. We then turn to the proof of Theorem 1.4 in Section 3. An appendix is devoted to the clarification of some properties of the Calderón-Zygmund decomposition.

## 2. Hardy inequalities

Before stating the Hardy inequalities required for the proof of Theorem 1.4 and for the convenience of the reader, we feel it is worthwile to write down a more self-contained proof of the $L^p$ Hardy inequality in the case where $M$ is a connected sum of the Euclidean spaces of dimension $\geq 2$. It is well-known that on $\mathbb{R}^n$ the following optimal Hardy inequality holds:

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{r^p} \leq \int_M |df|^p, \quad \forall f \in C^\infty(\mathbb{R}^n). \tag{$H_{\mathbb{R}^n}$}$$

Hence, the Hardy inequality on a connected sum of two Euclidean spaces follows from the following result, which we think is of interest by itself:

**Proposition 2.1.** *Let $M$ and $N$ be two Riemannian manifolds, such that $M$ and $N$ are isometric at infinity: there exists $K_M \Subset M$, $K_N \Subset N$ compact sets such that $M \setminus K_M$ is isometric to $N \setminus K_N$. Let $p \in (1, \infty)$. Then, the Hardy inequality*

$$\int_M \left(\frac{|f|}{r+1}\right)^p \lesssim \int_M |df|^p, \quad \forall f \in C_0^\infty(M). \tag{H}$$

*holds on $M$, if and only if it holds on $N$.*

**Corollary 2.2.** *Let $M = \mathbb{R}^n \sharp \mathbb{R}^n$ be a connected sum of two Euclidean spaces of dimension $n \geq 2$. Let $1 \leq p < n$. Then, $M$ satisfies the $L^p$ Hardy inequality (H).*

*Proof.* (of the proposition)

Assume that (H) holds on $N$. We are going to show that it holds on $M$ as well. By assumption, there exists two relatively compact, open sets $U \subset M$, $V \subset N$ such that $M \setminus U$ is isometric to $N \setminus V$. Let $0 \leq \chi \leq 1$ be a smooth, compactly supported function on $M$, which is equal to 1 identically in an neighborhood of $U$. Let $K \Subset M$ be a compact set containing the support of $\chi$. Let us take $f \in C_0^\infty(M)$, and write $f = \chi f + (1-\chi)f$. The function $(1-\chi)f$ identifies naturally with a smooth, compactly supported function defined on $N \setminus V$, hence the Hardy inequality on $N$ yields:

$$\int_N \left(\frac{|(1-\chi)f|}{1+r}\right)^p \lesssim \int_N |d\left((1-\chi)f\right)|^p.$$

Since $d\left((1-\chi)f\right) = -(d\chi)f + (1-\chi)df$, upon using the elementary inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ one gets:

$$\begin{aligned}\int_N \left(\frac{|(1-\chi)f|}{1+r}\right)^p &\lesssim \int_N |d\chi|^p |f|^p + \int_N |1-\chi|^p |df|^p \\ &\lesssim \int_K |f|^p + \int_M |df|^p\end{aligned}$$

On the other hand, one clearly has

$$\int_M \left(\frac{|\chi f|}{1+r}\right)^p \lesssim \int_K |f|^p,$$

so that, finally, one arrives to

$$\begin{aligned}\int_M \left(\frac{|f|}{1+r}\right)^p &\leq 2^{p-1}\int_M \left(\frac{|\chi f|}{1+r}\right)^p + 2^{p-1}\int_M \frac{|(1-\chi)f|^p}{r^p} \\ &\lesssim \int_K |f|^p + \int_M |df|^p\end{aligned}$$

Now, the assumed Hardy inequality on $N$ implies that $N$ is $p$-hyperbolic (see [8, Prop. 2.2]), and since $M$ and $N$ are isometric at infinity, it follows that the ends of $M$ are $p$-hyperbolic, hence $M$ itself is $p$-hyperbolic. For details, see [8, Section 2]. Therefore, there exists a constant $C_K$ such that for every $u \in C_0^\infty(M)$,

$$\int_K |u|^p \leq C_K \int_M |du|^p.$$

Combining this inequality with the previous one, one obtains that

$$\int_M \left(\frac{|f|}{1+r}\right)^p \lesssim \int_M |df|^p,$$

which is precisely the sought for Hardy inequality (H) on $M$.

□

Let us now state a more general result on $L^p$ Hardy inequalities that essentially stems from the work of V. Minerbe [18]:

**Theorem 2.3.** *Let $M$ be a complete Riemannian manifold satisfying* (D), (RCE) *and* (RD) *for an exponent $\nu > 1$. Let $1 \leq p < \nu$, and assume that* ($\mathrm{P}_p^E$) *holds. Then the $L^p$ Hardy inequality* (H) *holds on $M$.*

*Proof.* Note first that the (RCE) assumption implies that every end of $M$ satisfies the (RCA) condition considered in [18]. Next, (D) and ($\mathrm{P}_p^E$) implies that the proof of [18, Lemma 2.10], which provides $L^p$ Poincaré inequalities for subset of annuli, applies *mutatis mutandis* in our context. Given (RCA) in each end of $M$, one can then construct a "good covering" of $M$ (in the sense of [18, Definition 1.1]) for the pair of measure $(\frac{\mathrm{dvol}}{1+r^p}, \mathrm{dvol})$ as in [18, Section 2.3.1], and a weighted graph associated to this covering. The $L^p$ Poincaré inequalities for subset of annuli then implies that the good covering satisfies *continuous $L^p$ Sobolev inequalities of order $\infty$*, in the sense of [18, Definition 1.3]. In fact, an $L^p$ Sobolev inequalities of order $\infty$ is just another terminology for an $L^p$ Poincaré inequality. The proof of [18, Theorem 2.23] shows that the weighted graph satisfies an *isoperimetric inequality*. According to [18, Theorem 1.8], the continuous Sobolev inequality for the covering, together with the isoperimetric inequality for the weighted graph, imply the global $L^p$ Hardy inequality (H).

□

**Question 2.4.** Let $M$ be a complete Riemannian manifold satisfying (D), (UE), (RD) for an exponent $\nu > 1$, and ($\mathrm{P}_p^E$) for some $1 \leq p < \nu$; does the $L^p$ Hardy inequality (H) hold for $1 \leq p < \nu$ ? In other words, can the assumption (RCE) be replaced by (UE) in the statement of Theorem 2.3? For $p = 2$, it is proved in [16, Theorem 1.2] that, under (D), (RD) for some $\nu > 2$ and (UE), an $L^2$ Hardy inequality holds, however the proof does not extend easily to the case $p \neq 2$ unless one knows *a priori* that ($\mathrm{RR}_p$) holds (which of course we do not want to assume in the present paper).

## 3. Proof of the $L^p$ reverse inequality

To begin with, let us recall that, under the assumptions of Theorem 1.4, there exists a covering of $M$ by admissible balls, as well as an associated partition of unity. The following statement can be found in [9, Section 2.1]:

**Proposition 3.1.** *There exists a covering $(B_\alpha)_{\alpha\in\mathbb{N}}$ of $M$ by balls and an associated smooth partition of unity $(\chi_\alpha)_{\alpha\in\mathbb{N}}$ such that:*

1. *for every $\alpha \in \mathbb{N}$, the ball $B_\alpha$ is admissible,*
2. *the covering is* locally finite*: there exists $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}$,*

$$\mathrm{Card}\{\beta \in \mathbb{N}\,;\, B_\alpha \cap B_\beta \neq \emptyset\} \leq N,$$

(3) *for every* $R > 0$, *the set*

$$\{\alpha \in \mathbb{N}\,;\, B_\alpha \cap B(o,R) \neq \emptyset\}$$

*is finite,*

(4) *for all* $\alpha \in \mathbb{N}$, $0 \leq \chi_\alpha \leq 1$ *and* $\chi_\alpha$ *has support in* $B_\alpha$. *Moreover, there exists a constant* $C > 0$ *such that, for every* $\alpha \in \mathbb{N}$, $||\nabla \chi_\alpha||_\infty \leq \frac{C}{r_\alpha}$, *where* $r_\alpha$ *is the radius of* $B_\alpha$,

(5) *for all* $\alpha \neq 0$,

$$2^{-10} r(x_\alpha) \leq r_\alpha \leq 2^{-9} r(x_\alpha). \tag{3.1}$$

One can assume that $B_0 = B(o, r_0)$, and up to enlarging the value of $r_0$ and discarding a finite number of balls intersecting $B_0$, one can also assume that each of the remaining balls $B$ of the covering is such that $14B$ is remote. In the sequel, we thus assume that the balls have been relabeled in such a way that $B_0 = B(o, r_0)$ and for $\alpha \neq 0$, $14B_\alpha$ is remote.

Let us mention that point (5) of Proposition 3.1 will play an important role in the last part of the proof of Theorem 1.4 in which the Hardy inequality will be utilized. See Lemma 3.7. The assumptions of Theorem 1.4 imply that, for all $\alpha \in \mathbb{N}$, all balls inside $14B_\alpha$ support the $L^q$-Poincaré inequality; in particular, if $\tilde{B} \subset 2B_\alpha$, then $7\tilde{B}$ supports the $L^q$-Poincaré inequality.

The idea for the proof of Theorem 1.4 is as follows: first, decompose $f$ into

$$f = \sum_{\alpha \in \mathbb{N}} \chi_\alpha f =: \sum_{\alpha \in \mathbb{N}} f_\alpha.$$

We are going to estimate separately the "diagonal terms" $||\Delta^{1/2} f_\alpha||_{L^p(4B_\alpha)}$ and the "off-diagonal" terms $||\Delta^{1/2} f_\alpha||_{L^p(M \setminus 4B_\alpha)}$ for all $\alpha \in \mathbb{N}$.

3.1. **Estimates of the diagonal terms.** We first explain how to deal with the "diagonal" term $||\Delta^{1/2} f_\alpha||_{L^p(4B_\alpha)}$, using ideas from [2]. The main tool is a precise localized Calderón-Zygmund decomposition for gradients of functions, which is a variation on [2, Prop. 1.1]. Define the (uncentered) maximal function $\mathscr{M}$ by

$$\mathscr{M} u(x) = \sup_{x \ni B} \frac{1}{V(B)} \int_B |u| \, d\mu,$$

for all functions $u \in L^1_{loc}(M)$ and all $x \in M$. The required Calderón-Zygmund decomposition is as follows:

**Lemma 3.2.** *Let* $B$ *be a ball in* $M$, *and* $u \in C_0^\infty(B)$. *Let* $1 \leq q < \infty$, *and assume that, for all balls* $\widetilde{B} \subset 2B$, *the Poincaré inequality with exponent* $q$ *holds in* $7\tilde{B}$. *Then, there exists a constant* $C > 0$ *depending only on the*

*doubling constant, with the following property: for all* $\lambda > \left(\frac{C\,\|\nabla u\|_q^q}{V(B)}\right)^{\frac{1}{q}}$, *let*

$$\Omega := \{x \in M\,;\, \mathscr{M}(|\nabla u^q)(x) > \lambda^q\}.$$

*Then,* $\Omega \subset 2B$, *and there exists a denumerable collection of balls* $(B_i)_{i\geq 1} \subset \Omega \subset 2B$ *covering* $\Omega$, *a denumerable collection of* $C^1$ *functions* $(b_i)_{i\geq 1}$ *and a Lipschitz function* $g$ *such that:*

(1) $u = g + \sum_{i\geq 1} b_i$,

(2) *the support of* $g$ *is included in* $2B$, *and* $|\nabla g(x)| \lesssim \lambda$, *for a.e.* $x \in M$. *Moreover, there exists a bounded vector field* $H \in L^\infty(TM)$ *vanishing outside* $\Omega$, *such that*

$$\nabla g = \nabla u \cdot \mathbf{1}_{M\setminus\Omega} + H \ \text{a.e.}, \quad ||H||_\infty \lesssim \lambda, \tag{3.2}$$

(3) *the support of* $b_i$ *is included in* $B_i$,

$$\int_{B_i} |b_i|^q d\mu \lesssim r_i^q \int_{B_i} |\nabla u|^q \, d\mu,$$

*and*

$$\int_{B_i} |\nabla b_i|^q d\mu \lesssim \lambda^q V(B_i).$$

(4) $\sum_{i\geq 1} V(B_i) \lesssim \lambda^{-q} \int |\nabla u|^q d\mu$,

(5) *there is a finite upper bound* $N$ *for the number of balls* $B_i$ *that have a non-empty intersection,*

(6) *if* $B_i \cap B_j \neq \emptyset$ *and we denote by* $r_i$ *(reps.* $r_j$*) their radius, then*

$$\frac{1}{3} r_j \leq r_i \leq 3 r_j,$$

(7) *for every* $i \in \mathbb{N}$,

$$3B_i \cap (M \setminus \Omega) \neq \emptyset.$$

The construction of the covering and of the functions $(b_i)_{i\geq 1}$ has been explained in details in [9, Appendix B]; property 3 is an easy application of the Poincaré inequality which holds for every ball $B_i$. It turns out that property (2) is subtle, in fact the proof of Proposition 1.1 in [2] has a gap, which has subsequently been addressed in the unpublished note [1]. For the sake of clarification of this point, we provide a proof of points 2-4 from Lemma 3.2 in the Appendix.

Let us now turn to the estimate of $\left\|\Delta^{1/2} f_\alpha\right\|_{L^p(4B_\alpha)}$. Following [2], we first prove:

**Lemma 3.3.** *For all $\alpha \in \mathbb{N}$, all $\varphi \in C_c^\infty(B_\alpha)$ and all $\lambda > \left(\frac{C\,\|\nabla h\|_q^q}{V(B_\alpha)}\right)^{\frac{1}{q}}$,*

$$\mu(\{x \in 4B_\alpha;\ |\Delta^{1/2}\varphi(x)| > \lambda\}) \lesssim \frac{1}{\lambda^q} \int_{B_\alpha} |\nabla\varphi|^q \, d\mu. \tag{3.3}$$

*Proof.* For every $\lambda > \left(\frac{C\,\|\nabla\varphi\|_q^q}{V(B_\alpha)}\right)^{\frac{1}{q}}$, Lemma 3.2 provides a collection of balls $(B_\alpha^i)_{i\geq 1}$ included in $2B_\alpha$, a Lipschitz function $g_\alpha$ and a collection of $C^1$ functions $(b_\alpha^i)_{i\geq 1}$ sharing the properties listed in Lemma 3.2. In particular,

$$\varphi = g_\alpha + \sum_i b_\alpha^i.$$

Note that, for all $\alpha \in \mathbb{N}$ and all $i \geq 1$, $B_\alpha^i \subset 2B_\alpha$ and the balls $B_\alpha^i$ then satisfy the $L^q$ Poincaré inequality.
In the sequel of the argument, we use the following integral representation of $\Delta^{1/2}$:

$$\Delta^{1/2} = c \int_0^{+\infty} \Delta e^{-t\Delta} \frac{dt}{\sqrt{t}},$$

where $c > 0$ is an unimportant constant. As in [2, Section 1.2], it is enough to prove the required estimates for $\int_\varepsilon^R \Delta e^{-t\Delta} \frac{dt}{\sqrt{t}}$ for $0 < \varepsilon < R < \infty$, with constants independent of $\varepsilon, R$. In what follows, we ignore this issue and write directly $\int_0^{+\infty}$. The meaning of $\Delta^{1/2} g_\alpha$ and $\Delta^{1/2} b_\alpha^i$ is analogous to the one given in [2, Section 1.2] and relies on the pointwise Gaussian upper bounds (UE) and (1.2) for $p_t(x,y)$ and $\left|\frac{\partial p_t}{\partial t}(x,y)\right|$ respectively.
We first claim that

$$\mu\left(\left\{x \in 4B_\alpha;\ \left|\Delta^{1/2} g_\alpha(x)\right| > \frac{\lambda}{3}\right\}\right) \leq \frac{C}{\lambda^q} \int_{2B_\alpha} |\nabla g_\alpha(x)|^q \, d\mu(x). \tag{3.4}$$

Indeed,

$$\begin{aligned}
\mu\left(\left\{x \in 4B_\alpha;\ \left|\Delta^{1/2} g_\alpha(x)\right| > \frac{\lambda}{3}\right\}\right) &\leq \frac{9}{\lambda^2} \int_{4B_\alpha} \left|\Delta^{1/2} g_\alpha(x)\right|^2 d\mu(x) \\
&\leq \frac{9}{\lambda^2} \int_M |\nabla g_\alpha(x)|^2 \, d\mu(x) \\
&\lesssim \frac{1}{\lambda^2} \lambda^{2-q} \int_M |\nabla g_\alpha(x)|^q \, d\mu(x) \\
&\lesssim \frac{1}{\lambda^q} \int_M |\nabla\varphi(x)|^q \, d\mu(x).
\end{aligned}$$

The last line is due to the fact that

$$\begin{aligned}\int_M \left|\sum_i \nabla b_\alpha^i(x)\right|^q d\mu(x) &\lesssim \sum_i \int_M \left|\nabla b_\alpha^i(x)\right|^q d\mu(x)\\ &\lesssim \lambda^q \sum_i V(B_\alpha^i)\\ &\lesssim \int_M |\nabla\varphi(x)|^q \, d\mu(x),\end{aligned}$$

which implies in turn that

$$\begin{aligned}\|\nabla g_\alpha\|_q &\leq \|\nabla\varphi\|_q + \left\|\sum_i \nabla b_\alpha^i\right\|_q\\ &\lesssim \|\nabla\varphi\|_q .\end{aligned}$$

To cope with the terms involving $\Delta^{1/2} b_\alpha^i$, decompose

$$\begin{aligned}\Delta^{1/2} b_\alpha^i &= c\int_0^{+\infty} \Delta e^{-t\Delta} b_\alpha^i \frac{dt}{\sqrt{t}}\\ &= c\int_0^{(r_\alpha^i)^2} \Delta e^{-t\Delta} b_\alpha^i \frac{dt}{\sqrt{t}} + c\int_{(r_\alpha^i)^2}^{+\infty} \Delta e^{-t\Delta} b_\alpha^i \frac{dt}{\sqrt{t}}\\ &=: T_\alpha^i b_\alpha^i + U_\alpha^i b_\alpha^i. \end{aligned} \tag{3.5}$$

We therefore have to establish

$$I := \mu\left(\left\{x\in 4B_\alpha;\ \left|\sum_i T_\alpha^i b_\alpha^i(x)\right| > \frac{\lambda}{3}\right\}\right) \lesssim \frac{1}{\lambda^q}\|\nabla\varphi\|_q^q \tag{3.6}$$

and

$$J := \mu\left(\left\{x\in 4B_\alpha;\ \left|\sum_i U_\alpha^i b_\alpha^i(x)\right| > \frac{\lambda}{3}\right\}\right) \lesssim \frac{1}{\lambda^q}\|\nabla\varphi\|_q^q . \tag{3.7}$$

Let us first consider (3.6). The quantity $I$ is easily estimated by

$$\begin{aligned} I &\leq \mu\left(\bigcup_i 2B_\alpha^i\right) + \mu\left(\left\{x\in 4B_\alpha \setminus \bigcup_i 2B_\alpha^i;\ \left|\sum_i T_\alpha^i b_\alpha^i(x)\right| > \frac{\lambda}{3}\right\}\right)\\ &=: I_\alpha + J_\alpha.\end{aligned}$$

First, (D) and the properties of the Calderón-Zygmund decomposition yield at once

$$\begin{aligned} I_\alpha &\leq \sum_i V(2B_\alpha^i)\\ &\lesssim \sum_i V(B_\alpha^i)\\ &\lesssim \lambda^{-q}\int_M |\nabla\varphi(x)|^q \, d\mu(x).\end{aligned}$$

As far as $J_\alpha$ is concerned, one has

$$\begin{aligned} J_\alpha &\leq \frac{9}{\lambda^2}\int_{4B_\alpha\setminus\bigcup_i 2B_\alpha^i}\left|\sum_i T_\alpha^i b_\alpha^i(x)\right|^2 d\mu(x) \\ &\leq \frac{9}{\lambda^2}\int_{4B_\alpha}\left|\sum_i u_\alpha^i(x)\right|^2 d\mu(x), \end{aligned} \tag{3.8}$$

where

$$u_\alpha^i := \mathbf{1}_{4B_\alpha\setminus 2B_\alpha^i}\left|T_\alpha^i b_\alpha^i\right|.$$

To estimate the right-hand side in (3.8), we argue by duality. Pick up a fonction $v\in L^2(4B_\alpha)$ with $\|v\|_2=1$ and decompose

$$\begin{aligned} \left|\int_{4B_\alpha}\sum_i u_\alpha^i(x)v(x)d\mu(x)\right| &= \left|\sum_i\sum_{j\geq 1}\int_{C_j(B_\alpha^i)} u_\alpha^i(x)v(x)d\mu(x)\right| \\ &=: \left|\sum_i\sum_{j\geq 1} A_{ij}^\alpha\right|, \end{aligned} \tag{3.9}$$

where

$$C_j(B) := 2^{j+1}B\setminus 2^jB$$

for all open balls $B\subset M$ and all $j\geq 1$. In order to estimate $A_{ij}^\alpha$, we need a pointwise upper bound for $\left|\frac{\partial}{\partial t}e^{-t\Delta}b_\alpha^i\right|$ in $C_j(B_\alpha^i)$, $j\geq 1$. So, let $j\geq 1$ and $x\in C_j(B_\alpha^i)$. Denote by $x_\alpha^i$ the center of $B_\alpha^i$, and notice that (VD) and (1.1) imply, for all $z\in B_\alpha^i$ and all $t\in(0,(r_\alpha^i)^2)$,

$$\begin{aligned} \frac{V(x_\alpha^i,\sqrt{t})}{V(z,\sqrt{t})} &= \frac{V(x_\alpha^i,\sqrt{t})}{V(x_\alpha^i,r_\alpha^i)}\cdot\frac{V(x_\alpha^i,r_\alpha^i)}{V(z,r_\alpha^i)}\cdot\frac{V(z,r_\alpha^i)}{V(z,\sqrt{t})} \\ &\lesssim \left(\frac{r_\alpha^i}{\sqrt{t}}\right)^D. \end{aligned}$$

Bearing in mind that $b_\alpha^i$ has support in $B_\alpha^i$, that, for all $x\in C_j(B_\alpha^i)$ and all $z\in B_\alpha^i$, one has

$$d(x,z)\geq d(x,x_\alpha^i)-d(z,x_\alpha^i)\geq(2^j-1)r_\alpha^i\geq\frac{1}{2}2^j r_\alpha^i$$

(recall that $j\geq 1$) and using (1.3), one obtains, for all $x\in C_j(B_\alpha^i)$,

$$\begin{aligned}
\left|\frac{\partial}{\partial t}e^{-t\Delta}b_\alpha^i(x)\right| &\lesssim \frac{1}{t}\left(\frac{r_\alpha^i}{\sqrt{t}}\right)^D \frac{e^{-c\frac{4^j(r_\alpha^i)^2}{t}}}{V(x_\alpha^i,\sqrt{t})}\int_{B_\alpha^i}|b_\alpha^i(z)|\,d\mu(z)\\
&\lesssim \frac{1}{t}\left(\frac{r_\alpha^i}{\sqrt{t}}\right)^D e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\frac{V(x_\alpha^i,r_\alpha^i)}{V(x_\alpha^i,\sqrt{t})}\fint_{B_\alpha^i}|b_\alpha^i(z)|\,d\mu(z)\\
&\lesssim \frac{1}{t}\left(\frac{r_\alpha^i}{\sqrt{t}}\right)^{2D} e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\left(\fint_{B_\alpha^i}|b_\alpha^i(z)|^q\,d\mu(z)\right)^{1/q}\\
&\lesssim \frac{r_\alpha^i}{t}\left(\frac{(r_\alpha^i)^2}{t}\right)^D e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\frac{1}{(V(B_\alpha^i))^{1/q}}\|\nabla\varphi\|_{L^q(B_\alpha^i)}\\
&\leq \frac{r_\alpha^i}{t}\left(\frac{(r_\alpha^i)^2}{t}\right)^D e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\lambda,
\end{aligned}$$

where, in the third line, we have used (VD) and Hölder's inequality, while the fourth one follows from point (3) in Lemma 3.2. As a consequence, using doubling again, one obtains

$$\begin{aligned}
\left\|\Delta e^{-t\Delta}b_\alpha^i\right\|_{L^2(C_j(B_\alpha^i))} &\leq \mu(C_j(B_\alpha^i))^{1/2}\left\|\Delta e^{-t\Delta}b_\alpha^i\right\|_{L^\infty(C_j(B_\alpha^i))}\\
&\lesssim V(2^jB_\alpha^i)^{1/2}\frac{r_\alpha^i}{t}\left(\frac{(r_\alpha^i)^2}{t}\right)^D e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\lambda.
\end{aligned}$$

From thus, we infer that

$$\begin{aligned}
\left\|u_\alpha^i\right\|_{L^2(C_j(B_\alpha^i))} &= \left\|T_\alpha^i b_\alpha^i\right\|_{L^2(C_j(B_\alpha^i))}\\
&\leq \int_0^{(r_\alpha^i)^2}\left\|\Delta e^{-t\Delta}b_\alpha^i\right\|_{L^2(C_j(B_\alpha^i))}\frac{dt}{\sqrt{t}}\\
&\lesssim V(2^jB_\alpha^i)^{1/2}\lambda\int_0^{(r_\alpha^i)^2}\frac{r_\alpha^i}{t}\left(\frac{(r_\alpha^i)^2}{t}\right)^D e^{-c\frac{4^j(r_\alpha^i)^2}{t}}\frac{dt}{\sqrt{t}}\\
&\lesssim V(2^jB_\alpha^i)^{1/2}\lambda\int_0^1\frac{1}{u}\left(\frac{1}{u}\right)^D e^{-c\frac{4^j}{u}}\frac{du}{\sqrt{u}}\\
&\lesssim V(2^jB_\alpha^i)^{1/2}e^{-\frac{c}{2}4^j}\lambda,. \qquad (3.10)
\end{aligned}$$

where, in the fourth line, we made the change of.variables $t=(r_\alpha^i)^2u$.

On the other hand, for all $y\in B_\alpha^i$,

$$\begin{aligned}
\left(\int_{C_j(B_\alpha^i)}|v(z)|^2\,d\mu(z)\right)^{\frac12} &\leq \left(\int_{2^{j+1}B_\alpha^i}|v(z)|^2\,d\mu(z)\right)^{\frac12}\\
&\lesssim V^{1/2}(2^{j+1}B_\alpha^i)\left(\mathscr{M}(|v|^2)(y)\right)^{1/2}. \qquad (3.11)
\end{aligned}$$

Gathering (3.10) and (3.11) and using Cauchy-Schwarz and (D), one therefore obtains

$$\begin{aligned} A_{ij}^{\alpha} &\leq . \quad V(2^j B_\alpha^i)^{1/2} e^{-\frac{c}{2}4^j} \lambda \cdot V^{1/2}(B_\alpha^i) \fint_{B_\alpha^i} \left(\mathscr{M}(|v|^2)(y)\right)^{1/2} d\mu(y) \\ &\lesssim \quad 2^{jD/2} e^{-\frac{c}{2}4^j} \lambda \int_{B_\alpha^i} \left(\mathscr{M}(|v|^2)(y)\right)^{1/2} d\mu(y). \end{aligned}$$

Summing up over $i, j$ and recalling (3.9), one deduces

$$\begin{aligned} \left| \int_{4B_\alpha} \sum_i u_\alpha^i(x) v(x) d\mu(x) \right| &\lesssim \lambda \int_{4B_\alpha} \sum_i \mathbf{1}_{B_\alpha^i}(y) \left(\mathscr{M}(|v|^2)(y)\right)^{1/2} d\mu(y) \\ &\lesssim N\lambda \int_{\bigcup_i B_\alpha^i} \left(\mathscr{M}(|v|^2)(y)\right)^{1/2} d\mu(y) \\ &\lesssim N\lambda\mu \left(\bigcup_i B_\alpha^i\right)^{1/2} \left\| \left(\mathscr{M}(|v|^2)\right) \right\|_{1,\infty}^{1/2} \\ &\lesssim N\lambda\mu \left(\bigcup_i B_\alpha^i\right)^{1/2} \left\| |v|^2 \right\|_1 \\ &\lesssim N\lambda\mu \left(\bigcup_i B_\alpha^i\right)^{1/2}, \end{aligned} \tag{3.12}$$

where the second line follows from the finite overlap property for the balls $B_\alpha^i$ (recall that $N$ is given by Lemma 3.2), the third one is due to the Kolmogorov inequality ([17, Lemma 10, Section 7.7]) and the fourth one to the weak $(1,1)$ boundedness of $\mathscr{M}$. Finally, taking the supremum over all functions $v \in L^2(4B_\alpha)$ such that $\|v\|_{L^2(4B_\alpha)} = 1$ and recalling (3.8), we conclude

$$\begin{aligned} J_\alpha &\leq \frac{9}{\lambda^2} \int_{4B_\alpha} \left| \sum_i u_\alpha^i(x) \right|^2 d\mu(x) \\ &\lesssim \mu \left(\bigcup_i B_\alpha^i\right) \\ &\lesssim \frac{1}{\lambda^q} \|\nabla\varphi\|_q^q. \end{aligned}$$

Thus, (3.6) is proved.

Let us now turn to the proof of (3.7). We follow ideas in [2, Section 1.2], however we estimate the $L^q$ norm of $U_\alpha^i b_\alpha^i$ for each $\alpha$ separately, instead of considering the $L^q$-norm of $\sum_\alpha U_\alpha^i b_\alpha^i$ as in [2]. We write

$$U_\alpha^i b_\alpha^i = c \int_{(r_\alpha^i)^2}^{\infty} t\Delta e^{-t\Delta} \left(\frac{b_\alpha^i}{\sqrt{t}}\right) \frac{dt}{t} = c \int_0^{\infty} t\Delta e^{-t\Delta} b_t \frac{dt}{t},$$

with

$$b_t := \frac{b_\alpha^i}{\sqrt{t}} \mathbf{1}_{[(r_\alpha^i)^2, +\infty[}(t).$$

Let $g \in L^{q'}(4B_\alpha)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and $\|g\|_{L^{q'}(4B_\alpha)} = 1$. Since $q' \in (1, +\infty)$, Littlewood-Paley-Stein estimates ([20, Chapter 4, Theorem 10]) yield

$$\begin{aligned}
\left| \int_{4B_\alpha} (U_\alpha^i b_\alpha^i) g d\mu \right| &= \left| \int_0^\infty \langle t\Delta e^{-t\Delta} b_t, g \rangle \right| \\
&= \left| \int_0^\infty \langle b_t, t\Delta e^{-t\Delta} g \rangle \right| \\
&\leq \left\| \left( \int_0^\infty |b_t|^2 \frac{dt}{t} \right)^{1/2} \right\|_q \left\| \left( \int_0^\infty |t\Delta e^{-t\Delta} g|^2 \frac{dt}{t} \right)^{1/2} \right\|_{q'} \\
&\lesssim \left\| \left( \int_0^\infty |b_t|^2 \frac{dt}{t} \right)^{1/2} \right\|_q ||g||_{q'}.
\end{aligned}$$

It is easily seen that

$$\left\| \left( \int_0^\infty |b_t|^2 \frac{dt}{t} \right)^{1/2} \right\|_q = \frac{1}{r_\alpha^i} ||b_\alpha^i||_q,$$

hence

$$\left| \int_{4B_\alpha} (U_\alpha^i b_\alpha^i) g d\mu \right| \leq \frac{1}{r_\alpha} ||b_\alpha^i||_q \lesssim \|\nabla \varphi\|_{L^q(B_\alpha^i)},$$

where the last line is derived from point (3) in Lemma 3.2. Taking the supremum over all functions $g \in L^{q'}(4B_\alpha^i)$ with $\|g\|_{L^{q'}(4B_\alpha^i)} = 1$, we get

$$||U_\alpha^i b_\alpha^i||_{L^q(4B_\alpha)} \lesssim \|\nabla \varphi\|_{L^q(B_\alpha^i)}.$$

Summing up on $i$ and using the finite overlap property of the balls $B_\alpha^i$, one obtains

$$\begin{aligned}
\left\| \sum_i U_\alpha^i b_\alpha^i \right\|_{L^q(4B_\alpha)} &\lesssim \sum_i \left\| U_\alpha^i b_\alpha^i \right\|_{L^q(4B_\alpha)} \\
&\leq \sum_i \|\nabla \varphi\|_{L^q(B_\alpha^i)} \\
&\lesssim \|\nabla \varphi\|_{L^q(4B_\alpha)},
\end{aligned}$$

which entails at once that (3.7) holds. Gathering (3.4), (3.5), (3.6) and (3.7) concludes the proof of Lemma 3.3. □

As a consequence of the weak type estimate provided by Lemma 3.3, we are now going to prove:

**Lemma 3.4.** *Let $p \in (q,2)$. There is a constant $C>0$ such that for every $\alpha \in \mathbb{N}$,*

$$||\Delta^{1/2} f_\alpha||^p_{L^p(4B_\alpha)} \leq C \int_{B_\alpha} |\nabla f_\alpha|^p.$$

*Proof.* Let $C>0$ be given by Lemma 3.3. We first claim that

$$I := \int_{\left(\frac{C\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{1/p}}^{\infty} \lambda^{p-1} \mu(\{x \in 4B_\alpha;\ |\Delta^{1/2} f_\alpha(x)| > \lambda\})\, d\lambda \lesssim \int_{B_\alpha} |\nabla f_\alpha|^p \, d\mu. \tag{3.13}$$

This estimate will be established through an interpolation type argument borrowed from [2, Section 1.3]. Noticing that, since $f_\alpha$ is supported in $B_\alpha$ and by the Hölder inequality,

$$\left(\frac{\|\nabla f_\alpha\|_q^q}{V(B_\alpha)}\right)^{\frac{1}{q}} \leq \left(\frac{\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{\frac{1}{p}}$$

and using the Calderón-Zygmund decomposition given by Lemma 3.2 again for $\lambda > \left(\frac{C\|\nabla f_\alpha\|_q^q}{V(B_\alpha)}\right)^{1/q}$, $f_\alpha$ is decomposed as

$$f_\alpha =: g_\alpha + b_\alpha,$$

which yields

$$\begin{aligned}
I &\leq \int_{\left(\frac{C\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{1/p}}^{\infty} \lambda^{p-1} \mu\left(\left\{x \in 4B_\alpha;\ |\Delta^{1/2} g_\alpha(x)| > \frac{\lambda}{2}\right\}\right) d\lambda \\
&+ \int_{\left(\frac{C\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{1/p}}^{\infty} \lambda^{p-1} \mu\left(\left\{x \in 4B_\alpha;\ |\Delta^{1/2} b_\alpha(x)| > \frac{\lambda}{2}\right\}\right) d\lambda \\
&\lesssim \int_0^\infty \lambda^{p-1} \frac{|||\nabla g_\alpha|||_2^2}{\lambda^2} d\lambda \\
&+ \int_0^\infty \lambda^{p-1} \frac{|||\nabla b_\alpha|||_q^q}{\lambda^q} d\lambda \\
&=: I_1 + I_2.
\end{aligned}$$

In the fourth line, we used Lemma 3.3 with the function $b_\alpha$. Let us first estimate $I_1$. Lemma 3.2 yields

$$\nabla g_\alpha = \nabla f_\alpha \cdot \mathbf{1}_{M \setminus \Omega_\alpha} + h_\alpha$$

where $h_\alpha$ is supported in $\Omega_\alpha$ and $\|h_\alpha\|_\infty \lesssim \lambda$. This decomposition provides

$$\begin{aligned} I_1 &\lesssim \int_0^\infty \lambda^{p-1} \frac{\| |\nabla f_\alpha| \|^2_{L^2(4B_\alpha \setminus \Omega_\alpha)}}{\lambda^2} d\lambda \\ &+ \int_0^\infty \lambda^{p-1} \frac{\|h_\alpha\|_2^2}{\lambda^2} d\lambda \\ &=: I_1^1 + I_1^2. \end{aligned} \tag{3.14}$$

On the one hand, since $p < 2$,

$$\begin{aligned} I_1^1 &\leq \int_0^\infty \lambda^{p-3} \left( \int_{4B_\alpha \setminus \Omega_\alpha} |\nabla f_\alpha|^2 \, d\mu \right) d\lambda \\ &\leq \int_{4B_\alpha} |\nabla f_\alpha(x)|^2 \left( \int_{(\mathscr{M}(|\nabla f_\alpha|^q)(x))^{1/q}}^\infty \lambda^{p-3} d\lambda \right) d\mu(x) \\ &\simeq \int_{4B_\alpha} |\nabla f_\alpha(x)|^2 \left( \mathscr{M} \left( |\nabla f_\alpha|^q \right)(x) \right)^{\frac{p-2}{q}} d\mu(x), \end{aligned} \tag{3.15}$$

where, in order to pass from the first to the second line, we have used that by definition of $\Omega_\alpha$,

$$4B_\alpha \setminus \Omega_\alpha = \{ x \in 4B_\alpha \, ; \, (\mathscr{M} \left( |\nabla f_\alpha|^q \right)(x))^{1/q} \leq \lambda \}.$$

Since

$$\begin{aligned} |\nabla f_\alpha(x)|^2 &= |\nabla f_\alpha(x)|^p \, |\nabla f_\alpha(x)|^{2-p} \\ &\leq |\nabla f_\alpha(x)|^p \left( \mathscr{M} \left( |\nabla f_\alpha|^q \right)(x) \right)^{\frac{2-p}{q}} \quad \text{a. e. } x \in M, \end{aligned}$$

it follows from (3.15) that

$$I_1^1 \lesssim \int_{4B_\alpha} |\nabla f_\alpha(x)|^p \, d\mu(x) = \int_{B_\alpha} |\nabla f_\alpha(x)|^p \, d\mu(x) \tag{3.16}$$

(recall that $f_\alpha$ has support inside $B_\alpha$). On the other hand, since $h_\alpha$ is supported in $\Omega_\alpha$ and $|h_\alpha| \lesssim \lambda$, we get by using the definition of $\Omega_\alpha$ that

$$\begin{aligned} I_1^2 &\leq \int_0^\infty \lambda^{p-1}\mu(\Omega_\alpha)d\lambda \\ &= \int_0^\infty \lambda^{p-1}\left(\int_{4B_\alpha} \mathbf{1}_{\Omega_\alpha}(x)d\mu(x)\right)d\lambda \\ &\leq \int_{4B_\alpha}\left(\int_0^{(\mathscr{M}(|\nabla f_\alpha|^q)(x))^{1/q}} \lambda^{p-1}d\lambda\right)d\mu(x) \\ &\lesssim \int_{4B_\alpha} \left(\mathscr{M}\left(|\nabla f_\alpha|^q\right)(x)\right)^{p/q} d\mu(x) \\ &\leq \left\|\mathscr{M}\left(|\nabla f_\alpha|^q\right)\right\|_{p/q}^{p/q} \\ &\lesssim \left\||\nabla f_\alpha|^q\right\|_{p/q}^{p/q} \\ &= \int_{B_\alpha} |\nabla f_\alpha(x)|^p \, d\mu(x),. \end{aligned} \tag{3.17}$$

where the sixth line holds since $\frac{p}{q} > 1$. Gathering (3.14), (3.16) and (3.17) shows that

$$I_1 \lesssim \int_{B_\alpha} |\nabla f_\alpha(x)|^p \, d\mu(x). \tag{3.18}$$

Our next task is to estimate $I_2$. To that purpose, using Lemma 3.2 again, one starts from

$$\nabla b_\alpha = \nabla f_\alpha - \nabla g_\alpha = \nabla f_\alpha \cdot \mathbf{1}_{\Omega_\alpha} - h_\alpha,$$

which leads to

$$\begin{aligned} I_2 &\lesssim \int_0^\infty \lambda^{p-1}\frac{\||\nabla f_\alpha|\|_{L^q(\Omega_\alpha)}^q}{\lambda^q}d\lambda \\ &+ \int_0^\infty \lambda^{p-1}\frac{\|h_\alpha\|_{L^q(\Omega_\alpha)}^q}{\lambda^q}d\lambda \\ &=: I_2^1 + I_2^2. \end{aligned}$$

For $I_2^1$, one has, arguing as before,

$$\begin{aligned} I_2^1 &\leq \int_0^\infty \lambda^{p-q-1}\left(\int_{\Omega_\alpha} |\nabla f_\alpha(x)|^q \, d\mu(x)\right)d\lambda \\ &\leq \int_{4B_\alpha} |\nabla f_\alpha(x)|^q \left(\int_0^{(\mathscr{M}(|\nabla f_\alpha|^q)(x))^{1/q}} \lambda^{p-q-1}d\lambda\right)d\mu(x) \\ &\lesssim \int_{4B_\alpha} |\nabla f_\alpha(x)|^q \left(\mathscr{M}\left(|\nabla f_\alpha|^q\right)(x)\right)^{\frac{p-q}{q}} d\mu(x) \\ &\lesssim \left(\int_{4B_\alpha} |\nabla f_\alpha(x)|^p\right)^{\frac{q}{p}} \left(\int_{4B_\alpha} \left(\mathscr{M}\left(|\nabla f_\alpha|^q\right)(x)\right)^{\left(\frac{p}{q}\right)'\frac{p-q}{q}} d\mu(x)\right)^{1-\frac{q}{p}} \end{aligned}$$

where $\left(\frac{p}{q}\right)'$ is such that $\frac{q}{p}+\left[\left(\frac{p}{q}\right)'\right]^{-1}=1$. Since

$$\begin{aligned}\left(\left(\frac{p}{q}\right)'\frac{p-q}{q}\right)^{-1} &= \left(1-\frac{q}{p}\right)\frac{q}{p-q}\\ &= \frac{p-q}{p}\frac{q}{p-q}=\frac{q}{p},\end{aligned}$$

one therefore concludes, using the $L^{\frac{p}{q}}$-boundedness of $\mathscr{M}$,

$$\begin{aligned}I_2^1 &\lesssim \left(\int_{4B_\alpha}|\nabla f_\alpha(x)|^p\,d\mu(x)\right)^{\frac{q}{p}}\left(\int_{4B_\alpha}\left(\mathscr{M}\left(|\nabla f_\alpha|\right)^q(x)\right)^{\frac{p}{q}}d\mu(x)\right)^{1-\frac{q}{p}}\\ &\lesssim \left(\int_{4B_\alpha}|\nabla f_\alpha(x)|^p\,d\mu(x)\right)^{\frac{q}{p}}\left(\int_{4B_\alpha}|\nabla f_\alpha|^p(x)d\mu(x)\right)^{1-\frac{q}{p}}\\ &= \int_{4B_\alpha}|\nabla f_\alpha(x)|^p\,d\mu(x)\\ &= \int_{B_\alpha}|\nabla f_\alpha(x)|^p\,d\mu(x).\end{aligned}$$

The estimate of $I_2^2$ is analogous to the one of $I_1^2$, which concludes the proof of (3.13).

Let us consider now

$$J:=\int_0^{\left(\frac{C\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{1/p}}\lambda^{p-1}\mu(\{x\in 4B_\alpha\,;\,|\Delta^{1/2}f_\alpha(x)|>\lambda\})\,d\lambda.$$

Using a trivial estimate and doubling, one obtains at once

$$\mu(\{x\in 4B_\alpha\,;\,|\Delta^{1/2}f_\alpha(x)|>\lambda\})\leq V(4B_\alpha)\lesssim V(B_\alpha),$$

so that

$$J\lesssim V(B_\alpha)\int_0^{\left(\frac{C\|\nabla f_\alpha\|_p^p}{V(B_\alpha)}\right)^{1/p}}\lambda^{p-1}\,d\lambda=C\int_{B_\alpha}|\nabla f_\alpha|^p. \tag{3.19}$$

Putting together (3.13) and (3.19), we get

$$\begin{aligned}||\Delta^{1/2}f_\alpha||^p_{L^p(4B_\alpha)} &= p\int_0^\infty\lambda^{p-1}\mu(\{x\in 4B_\alpha\,;\,|\Delta^{1/2}f_\alpha(x)|>\lambda\})\,d\lambda\\ &\leq p(I+J)\\ &\lesssim \int_{B_\alpha}|\nabla f_\alpha|^p,\end{aligned}$$

which ends the proof of Lemma 3.4. □

3.2. **Estimates of the non-diagonal terms.** Let us now estimate the $L^p$ norm of $\Delta^{1/2} f_\alpha$ outside $4B_\alpha$. As before, we use the splitting

$$\Delta^{1/2} f_\alpha = \int_0^{r_\alpha^2} \frac{\partial}{\partial t} e^{-t\Delta} f_\alpha \, \frac{dt}{\sqrt{t}} + \int_{r_\alpha^2}^{\infty} \frac{\partial}{\partial t} e^{-t\Delta} f_\alpha \, \frac{dt}{\sqrt{t}} = T_\alpha f_\alpha + U_\alpha f_\alpha.$$

The term $U_\alpha f_\alpha$ is easily estimated:

**Lemma 3.5.** *Let $s \in (1, +\infty)$. Then, for every $\alpha \in \mathbb{N}$,*

$$||U_\alpha f_\alpha||_s \leq \frac{1}{r_\alpha} ||f_\alpha||_s.$$

In particular, this implies that

$$||U_\alpha f_\alpha||_{L^s(M \setminus 4B_\alpha)} \leq \frac{1}{r_\alpha} ||f_\alpha||_s.$$

*Proof.* We follow ideas in [2, Section 1.2] again, arguing as in the estimate of (3.7). We write

$$U_\alpha f_\alpha = \int_{r_\alpha^2}^{\infty} t\Delta e^{-t\Delta} \left( \frac{f_\alpha}{\sqrt{t}} \right) \frac{dt}{t} = \int_0^{\infty} t\Delta e^{-t\Delta} f_t \, \frac{dt}{t},$$

with

$$f_t = \frac{f_\alpha}{\sqrt{t}} \mathbf{1}_{[r_\alpha^2, +\infty[}(t).$$

Using duality and Littlewood-Paley-Stein estimates again, we obtain, analogously to the proof of (3.7),

$$||U_\alpha f_\alpha||_s \leq \frac{1}{r_\alpha} ||f_\alpha||_s.$$

□

Let is now turn to the terms $T_\alpha f_\alpha$:

**Lemma 3.6.** *Under the assumptions (D) and (UE), there exists a constant $C > 0$ such that for every $s \in [1, \infty)$ and every $\alpha \in \mathbb{N}$,*

$$||T_\alpha f_\alpha||_{L^s(M \setminus 4B_\alpha)} \leq \frac{C}{r_\alpha} ||f_\alpha||_s.$$

*Proof.* The argument is reminiscent of the one for (3.6). Let $\alpha \in \mathbb{N}$, and $0 < t < r_\alpha^2$. We first estimate $\left| \frac{\partial}{\partial t} e^{-t\Delta} f_\alpha \right|$ pointwise on $C_j^\alpha := C^j(B_\alpha) = 2^{j+1} B_\alpha \setminus 2^j B_\alpha$, $j \geq 2$. Let $j \geq 2$ and $x \in C_\alpha^j$. As before, (VD) and (1.1) imply, for all $z \in B_\alpha$,

$$\frac{V(x_\alpha, \sqrt{t})}{V(z, \sqrt{t})} \lesssim \left( \frac{r_\alpha}{\sqrt{t}} \right)^D.$$

Since $f_\alpha$ has support in $B_\alpha$ and (1.3) holds, one obtains, for all $x \in C_j^\alpha$, $j \geq 2$,

$$\begin{aligned}\left|\frac{\partial}{\partial t}e^{-t\Delta}f_\alpha(x)\right| &\lesssim \frac{1}{t}\left(\frac{r_\alpha}{\sqrt{t}}\right)^{2D} e^{-c\frac{4^j r_\alpha^2}{t}} \left(⨍_{B_\alpha} |f_\alpha(z)|^s \, d\mu(z)\right)^{1/s} \\ &= \frac{1}{r_\alpha}\left(\frac{r_\alpha^2}{t}\right)^{D+1} e^{-c\frac{4^j r_\alpha^2}{t}} \left(⨍_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, d\mu(z)\right)^{1/s}.\end{aligned}$$

Note that the condition $j \geq 2$ was used in the last inequality. As a consequence,

$$\begin{aligned}\left\|\Delta e^{-t\Delta}f_\alpha\right\|_{L^s(C_\alpha^j)} &\leq \left(\mu(C_\alpha^j)\right)^{1/s} \left\|\Delta e^{-t\Delta}f_\alpha\right\|_{L^\infty(C_\alpha^j)} \\ &\lesssim \left(V(2^{j+1}B_\alpha)\right)^{1/s} \frac{1}{r_\alpha}\left(\frac{r_\alpha^2}{t}\right)^{D+1} \frac{e^{-c\frac{4^j r_\alpha^2}{t}}}{(V(B_\alpha))^{1/s}} \left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, d\mu(z)\right)^{1/s} \\ &\lesssim \frac{1}{r_\alpha} 2^{jD/s}\left(\frac{r_\alpha^2}{t}\right)^{D+1} e^{-c\frac{4^j r_\alpha^2}{t}} \left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, d\mu(z)\right)^{1/s} \\ &\leq \frac{1}{r_\alpha}\left(\frac{2^j r_\alpha^2}{t}\right)^{D+1} e^{-c\frac{4^j r_\alpha^2}{t}} \left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, d\mu(z)\right)^{1/s} \\ &\leq \frac{1}{r_\alpha} e^{-c'\frac{4^j r_\alpha^2}{t}} \left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, d\mu(z)\right)^{1/s}.\end{aligned}$$

It follows that

$$\begin{aligned}\|T_\alpha f_\alpha\|_{L^s(C_\alpha^j)} &\lesssim \frac{1}{r_\alpha}\left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, dz\right)^{1/s} \int_0^{r_\alpha^2} e^{-c'\frac{4^j r_\alpha^2}{t}} \frac{dt}{\sqrt{t}} \\ &\leq \frac{1}{r_\alpha}\left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, dz\right)^{1/s} 2^j r_\alpha \left(\int_{4^j}^{+\infty} e^{-c'u} u^{-\frac{3}{2}} du\right) \\ &\lesssim e^{-c''2^j}\left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, dz\right)^{1/s},\end{aligned}$$

where, in the second line, we made the change of variables $u = \frac{4^j r_\alpha^2}{t}$. Therefore,

$$\begin{aligned}\|T_\alpha f_\alpha\|_{L^s(M\setminus 4B_\alpha)} &\leq \sum_{j\geq 2} \|T_\alpha f_\alpha\|_{L^s(C_\alpha^j)} \\ &\leq \left(\sum_{j\in\mathbb{N}} e^{-c''2^j}\right)\left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, dz\right)^{1/s} \\ &\lesssim \left(\int_{B_\alpha} \frac{|f_\alpha(z)|^s}{r_\alpha^s} \, dz\right)^{1/s}.\end{aligned}$$

□

Summarizing what we have done so far, we get, according to Lemmas 3.4, 3.5 and 3.6:

$$\begin{aligned}
||\Delta^{1/2} f||_{L^p(M)} &= ||\Delta^{1/2} \sum_{\alpha\in\mathbb{N}} f_\alpha||_p \\
&\leq \sum_{\alpha\in\mathbb{N}} ||\Delta^{1/2} f_\alpha||_{L^p(4B_\alpha)} + \sum_{\alpha\in\mathbb{N}} ||\Delta^{1/2} f_\alpha||_{L^p(M\setminus 4B_\alpha)} \\
&\leq \sum_{\alpha\in\mathbb{N}} ||\Delta^{1/2} f_\alpha||_{L^p(4B_\alpha)} + \sum_{\alpha\in\mathbb{N}} ||T_\alpha f_\alpha||_{L^p(M\setminus 4B_\alpha)} \\
&\quad + \sum_{\alpha\in\mathbb{N}} ||U_\alpha f_\alpha||_{L^p(M\setminus 4B_\alpha)} \\
&\lesssim \sum_{\alpha\in\mathbb{N}} ||\nabla f_\alpha||_{L^p(4B_\alpha)} + \sum_{\alpha\in\mathbb{N}} \left\| \frac{f_\alpha}{r_\alpha} \right\|_{L^p}.
\end{aligned}$$

Recalling that $f_\alpha = \chi_\alpha f$ and $||\nabla\chi_\alpha||_\infty \lesssim \frac{1}{r_\alpha}$, one has

$$||\nabla f_\alpha||_p \lesssim \left\| \frac{f_\alpha}{r_\alpha} \right\|_p + ||\nabla f||_{L^p(B_\alpha)}.$$

Since the balls $(B_\alpha)_{\alpha\in\mathbb{N}}$ have the finite intersection property, one has

$$\sum_{\alpha\in\mathbb{N}} ||\nabla f||_{L^p(B_\alpha)} \lesssim ||\nabla f||_p.$$

Therefore,

$$||\Delta^{1/2} f||_{L^p(M)} \lesssim ||\nabla f||_p + \sum_{\alpha\in\mathbb{N}} \left\| \frac{f_\alpha}{r_\alpha} \right\|_{L^p}. \tag{3.20}$$

We now rely on the $L^p$ Hardy inequality to establish:

**Lemma 3.7.** *For all $p \in [q, 2)$, one has*

$$\sum_\alpha \left\| \frac{|f_\alpha|}{r_\alpha} \right\|_p \lesssim \|\nabla f\|_p.$$

*Proof.* Notice first that $(\mathrm{P}_p^E)$ holds, which entails, by Theorem 2.3, that the $L^p$ Hardy inequality

$$\int_M \left( \frac{|f|}{1+r} \right)^p d\mu \lesssim \int_M |\nabla f|^p d\mu$$

holds on $M$. Let $\alpha \in \mathbb{N}$. Then

$$|f_\alpha| \leq |f|\, \mathbf{1}_{B_\alpha},$$

and, for all $x \in B_\alpha$,

$$\frac{1}{r_\alpha} \lesssim \frac{1}{r(x)+1},$$

which is easily checked, whether $B_\alpha$ is anchored or remote (note that (3.1) is used in that case). Thus, using the finite overlap property for the balls $(B_\alpha)_{\alpha\in\mathbb{N}}$ again, one obtains

$$\begin{aligned}\sum_\alpha \left\| \frac{|f_\alpha|}{r_\alpha} \right\|_p &\lesssim \left( \int_M \frac{|f(x)|^p}{(r(x)+1)^p} d\mu(x) \right)^{\frac{1}{p}} \\ &\lesssim \|\nabla f\|_p ,\end{aligned}$$

where the last inequality follows from the Hardy inequality (H) (see Theorem 2.3). □

Finally, combining (3.20) and Lemma 3.7, we conclude that $(RR_p)$ holds, which concludes the proof of Theorem 1.4.

## 4. Appendix: proof of the Calderón-Zygmund lemma for Sobolev functions

In this section we explain the proof of Lemma 3.2; the construction of the (Whitney type) covering $(B_i)_{i\in\mathbb{N}}$ satisfying (5), (6) and (7), and of the functions $b_i$, has already been presented in details in [9, Appendix B]. Here we intend to explain mainly the proof of points 2-4. We assume also that $\Omega \neq \emptyset$, otherwise the Calderón-Zygmund decomposition simply writes $g = u$. We denote by $F$ the complement of $\Omega$ in $M$. The proof of (4) is easy enough: according to (5), one has

$$\sum_{i\geq 1} V(B_i) \leq N\mu(\Omega) \lesssim \lambda^{-q} \int |\nabla u|^q d\mu,$$

where in the last inequality we have used the weak (1,1) type of the maximal function and the definition of $\Omega$. This proves (4). Let us now recall how the functions $b_i$ are defined: according to [9], one can find a smooth partition of unity $(\chi_i)_{i\in\mathbb{N}}$ associated with the covering $(B_i)_{i\in\mathbb{N}}$ of $\Omega$, and such that for every $i \in \mathbb{N}$,

$$||\nabla \chi_i||_{L^\infty} \lesssim \frac{1}{r_i},$$

where $r_i$ denotes the radius of $B_i$. Then, $b_i$ is defined by

$$b_i = (u - u_{B_i})\chi_i.$$

It is clear by definition that $b_i$ has support in $B_i$. Moreover, since $B_i \subset \Omega \subset 2B$ (see [9]), it follows that $B_i$ satisfies the $L^q$ Poincaré inequality. Hence,

$$||b_i||_q \leq \left( \int_{B_i} |u - u_{B_i}|^q \, d\mu \right)^{1/q} \lesssim r_i ||\nabla u||_{L^q(B_i)}. \tag{4.1}$$

Also,

$$\nabla b_i = (u - u_{B_i})\nabla \chi_i + \chi_i \nabla u,$$

so that, again applying Poincaré on $B_i$ and the estimate on $\nabla\chi_i$, we obtain

$$||\nabla b_i||_q \lesssim ||\nabla u||_{L^q(B_i)}.$$

But property (7) in Lemma 3.2 and doubling imply that

$$\begin{aligned} ||\nabla u||^q_{L^q(B_i)} &\leq ||\nabla u||^q_{L^q(3B_i)} \\ &\leq V(3B_i)\lambda^q \\ &\lesssim V(B_i)\lambda^q, \end{aligned}$$

so (3) holds. Define

$$b = \sum_{i\geq 0} b_i,$$

and let

$$g = u - b.$$

We will see in a moment that $b$ is actually a well-defined, locally integrable function on $M$. Since $u$ has support in $B$ and $b$ in $2B$, it follows that $g$ has support in $2B$. It remains to prove (2). Since the covering is locally finite by (5), the sum defining $b$ is merely a finite sum at every point in $\Omega$. There is a subtle point which is that it is possible that the balls $B_i$ accumulate near the boundary of $\Omega$, making $\nabla b$ having a singularity on the boundary of $\Omega$ (think of the extreme case where $b = \mathbf{1}_\Omega$, for instance). So, despite the fact that each $b_i$ is smooth and has support inside $\Omega$, and despite the sum $\sum_{i\geq 0} b_i$ being locally finite in $\Omega$, one must check carefully that $b$ is Lipschitz up to the boundary of $\Omega$. First, let us see that the series defining $b$ converges in $L^1_{loc}(M)$. Indeed, let $K$ be a compact set in $M$, and $\varphi \in L^\infty(M)$ vanishing outside of $K$; then, for every $n \in \mathbb{N}$,

$$\begin{aligned} \langle \sum_{i\leq n} |b_i|, \varphi\rangle &= \sum_{i\leq n} \langle |b_i|, \varphi\rangle \\ &= \sum_{i\leq n} \langle \frac{|b_i|}{r_i}, r_i\varphi\rangle \\ &\leq \sum_{i\leq n} \langle \frac{|b_i|}{r_i}, r_i|\varphi|\rangle \\ &\leq \sum_{i\leq n} \left|\left|\frac{b_i}{r_i}\right|\right|_q \sup_{x\in K} d(x,F)||\varphi||_{L^{q'}} \\ &\lesssim \sum_{i\leq n} ||\nabla u||_{L^q(B_i)}||\varphi||_\infty \\ &\lesssim N||\nabla u||_q||\varphi||_\infty. \end{aligned}$$

Since $n$ is arbitrary, this proves that $\sum_{i\geq 0}|b_i|$ converges in $L^1_{loc}$, hence $b\in L^1_{loc}$ is well-defined. This yields that $g\in L^1_{loc}(M)$, too. The estimate on $||\nabla b_i||_q$ and the fact that the covering satisfies (4) in Lemma 3.2 easily imply that $\nabla b$, defined as a distribution, actually belongs to $L^q(M)$, and one has the following equality in $L^q$:

$$\nabla b=\sum_{i\geq 0}\nabla b_i=\sum_{i\geq 0}\left((u-u_{B_i})\nabla\chi_i+(\nabla u)\cdot\chi_i\right).$$

It is clear that $\sum_{i\geq 0}(\nabla u)\cdot\chi_i$ converges in $L^q$ to $(\nabla u)\cdot\mathbf{1}_\Omega$, hence $\nabla g$, defined as a distribution, actually belongs to $L^q(M)$ and we get the following equality in $L^q(M)$:

$$\nabla g=(\nabla u)\cdot\mathbf{1}_F-\sum_{i\geq 0}(u-u_{B_i})\nabla\chi_i.$$

Define

$$H=-\sum_{i\geq 0}(u-u_{B_i})\nabla\chi_i,$$

which is an $L^q$ vector field since the series of the $L^q$ norms converge. We claim that the vector field $H$ is in fact essentially bounded, and that we have the estimate $||H||_{L^\infty}\lesssim\lambda$. This is proven in [1] and we partially reproduce the proof from there, adding some more details.
Since $L^1(TM)\cap L^{q'}(TM)$ is dense in $L^1(TM)$, it is enough to prove that for every vector field $X\in L^1\cap L^{q'}(TM)$,

$$|\langle H,X\rangle|\lesssim\lambda||X||_1.$$

Here, $q'$ denotes the conjugate exponent to $q$, that is $\frac{1}{q}+\frac{1}{q'}=1$.
Fix such a vector field $X$, then by $L^q-L^{q'}$ duality,

$$\langle H,X\rangle=\lim_{n\to\infty}\int\left(\sum_{i\leq n}(u-u_{B_i})\cdot(\nabla\chi_i)\cdot X\right)d\mu.$$

Here, $\nabla\chi_i(x)\cdot X(x)$ denotes the inner product on the tangent space $T_xM$ defined by the Riemannian metric. Since $\sum_{i\leq n}(u-u_{B_i})\nabla\chi_i$ is a finite sum, it defines a smooth function with compact support inside $\Omega$. Using that $\sum_{m\in\mathbb{N}}\chi_m=\mathbf{1}_\Omega$, we have

$$\sum_{i\leq n}(u-u_{B_i})(\nabla\chi_i)\cdot X=\sum_{m\in\mathbb{N}}\sum_{i\leq n}(u-u_{B_i})(\nabla\chi_i)\cdot(\chi_m X).$$

Denote $X_m:=\chi_m X$, which has now compact support in $\Omega$. Denote by $I_m$ the set of indices $i$ for which $B_i\cap B_m\neq\emptyset$, which is a finite set of cardinal at most $N$ by (5). Then,

$$\begin{aligned}\sum_{i\leq n}(u-u_{B_i})(\nabla\chi_i)X_m &= \sum_{i\in I_m,\, i\leq n}(u-u_{B_i})(\nabla\chi_i)X_m \\ &= \sum_{i\in I_m,\, i\leq n}(u-u_{B_m})(\nabla\chi_i)X_m \\ &\quad + \sum_{i\in I_m,\, i\leq n}(u_{B_m}-u_{B_i})(\nabla\chi_i)X_m\end{aligned}$$

We deal with the first sum in the right hand side: using (6),

$$\begin{aligned}\sum_m \sum_{i\in I_m,\, i\leq n}|u-u_{B_m}|\cdot|\nabla\chi_i|\cdot|X_m| &\leq \sum_m\sum_{i\in I_m}|u-u_{B_m}|\cdot|\nabla\chi_i|\cdot|X_m| \\ &\lesssim \sum_m\sum_{i\in I_m}\frac{1}{r_i}|u-u_{B_m}|\chi_m\cdot|X| \\ &\lesssim N\sum_m\frac{1}{r_m}|u-u_{B_m}|\chi_m\cdot|X|\end{aligned}$$

Integrating the above inequality, using Fubini-Tonelli and Poincaré on each ball $B_m$ and (5), we get

$$\begin{aligned}\int\sum_{\substack{(m,i)\in\mathbb{N}^2\\ i\in I_m,\, i\leq n}}|u-u_{B_m}|\cdot|\nabla\chi_i|\cdot|X_m|\,d\mu &\leq N\sum_m\int\frac{1}{r_m}|u-u_{B_m}|\cdot\chi_m\cdot|X|\,d\mu \\ &\leq N\sum_m||r_m^{-1}(u-u_{B_m}))||_{L^q(B_m)}||X||_{L^{q'}} \\ &\lesssim N\sum_m||\nabla u||_{L^q(B_m)}||X||_{L^{q'}} \\ &\lesssim N^2||\nabla u||_{L^q}\cdot||X||_{L^{q'}}<+\infty\end{aligned}$$

Since the constant $N^2$ is independant of $n$, the limit

$$\sum_{\substack{(m,i)\in\mathbb{N}^2\\ i\in I_m}}(u-u_{B_m})(\nabla\chi_i)\cdot X_m=\lim_{n\to\infty}\sum_{\substack{(m,i)\in\mathbb{N}^2\\ i\in I_m,\, i\leq n}}(u-u_{B_m})(\nabla\chi_i)\cdot X_m$$

exists in $L^1$, and and one can evaluate it using Fubini to exchange the order of summation; since $\sum_{i\in\mathbb{N}}\chi_i=\mathbf{1}_\Omega$, we have by definition of $I_m$ that for every $m\in\mathbb{N}$, $\sum_{i\in I_m}\chi_i=1$ in restriction to $B_m$. Thus,

$$\begin{aligned}\sum_{i\in I_m}(\nabla\chi_i)\,\chi_m &= \chi_m\nabla\left(\sum_{i\in I_m}\chi_i\right)\\ &= \chi_m(\nabla\mathbf{1})\\ &= 0.\end{aligned}$$

Therefore, we have the following equality which holds in $L^1$:

$$\sum_{\substack{(m,i)\in\mathbb{N}^2\\ i\in I_m}}(u-u_{B_m})(\nabla\chi_i)\cdot(\chi_m X)=0.$$

Consequently, we have

$$-\langle H,X\rangle=\lim_{n\to\infty}\int\left(\sum_{\substack{(m,i)\in\mathbb{N}^2\\ i\in I_m,\,i\le n}}(u_{B_m}-u_{B_i})(\nabla\chi_i)\cdot X_m\right)d\mu.$$

We now estimate $|u_{B_m}-u_{B_i}|$ for $i\in I_m$. According to (6), $r_i\le 3r_m$, and since $B_i\cap B_m\neq\emptyset$, we have $B_i\subset 7B_m$. Also, since $B_m\subset 2B$, the $L^q$ Poincaré inequality holds for $7B_m$. We now estimate

$$\begin{aligned}|u_{7B_m}-u_{B_i}| &\le \int_{B_i}|u(x)-u_{7B_m}|\,\frac{d\mu(x)}{V(B_i)}\\ &\le \left(\int_{B_i}|u(x)-u_{7B_m}|^q\,\frac{d\mu(x)}{V(B_i)}\right)^{1/q}\\ &\le \left(\int_{7B_m}|u(x)-u_{7B_m}|^q\,\frac{d\mu(x)}{V(B_i)}\right)^{1/q}\\ &\lesssim r_m\left(\int_{7B_m}|\nabla u(x)|^q\,\frac{d\mu(x)}{V(B_m)}\right)^{1/q},\end{aligned}$$

where in the last line we have used doubling and the fact that $r_i\simeq r_m$. A completely analogous argument gives

$$|u_{7B_m}-u_{B_m}|\lesssim r_m\left(\int_{7B_m}|\nabla u(x)|^q\,\frac{d\mu(x)}{V(B_m)}\right)^{1/q},$$

and summing these two estimates we find that

$$|u_{B_m}-u_{B_i}|\lesssim r_m\left(\int_{7B_m}|\nabla u(x)|^q\,\frac{d\mu(x)}{V(B_m)}\right)^{1/q}.$$

Given that $|\nabla\chi_i|\lesssim\frac{1}{r_i}\lesssim\frac{1}{r_m}$, one obtains

$$|u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \lesssim \left( \int_{7B_m} |\nabla u(x)|^q \, \frac{d\mu(x)}{V(B_m)} \right)^{1/q}.$$

However, (7) entails that $7B_m \cap F \neq \emptyset$. The definition of $F$ in terms of the maximal function gives that

$$\int_{7B_m} |\nabla u(x)|^q \, \frac{d\mu(x)}{V(B_m)} \leq \lambda^q,$$

hence

$$|u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \lesssim \lambda.$$

It follows that

$$\begin{aligned} \sum_{\substack{(m,i)\in\mathbb{N}^2 \\ i\in I_m,\, i\leq n}} |u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \cdot |X_m| &\lesssim N\lambda \left( \sum_m \chi_m \right) |X| \\ &\lesssim \lambda |X|. \end{aligned}$$

Therefore, integrating one finds

$$\int \sum_{\substack{(m,i)\in\mathbb{N}^2 \\ i\in I_m,\, i\leq n}} |u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \cdot |X_m| d\mu \lesssim \lambda ||X||_1.$$

One thus concludes that

$$\int \sum_{\substack{(m,i)\in\mathbb{N}^2 \\ i\in I_m}} |u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \cdot |X_m| d\mu \lesssim \lambda ||X||_1.$$

But one has

$$|\langle H, X \rangle| \leq \int \sum_{\substack{(m,i)\in\mathbb{N}^2 \\ i\in I_m}} |u_{B_m} - u_{B_i}| \cdot |\nabla \chi_i| \cdot |X_m| d\mu,$$

which finally yields

$$|\langle H, X \rangle| \lesssim \lambda ||X||_1.$$

This estimate being valid for every $X \in L^1(TM) \cap L^{q'}(TM)$, which is dense in $L^1(TM)$, we conclude by duality that $H \in L^\infty$ with $||H||_{L^\infty} \lesssim \lambda$.

**Remark 4.1.** Note that the proof actually yields the following representation for $H$:

$$H = \sum_{\substack{(m,i)\in\mathbb{N}^2 \\ i\in I_m}} (u_{B_i} - u_{B_m}) \cdot \chi_m \cdot (\nabla \chi_i) \quad \text{a.e.},$$

the right-hand side being an essentially bounded vector field with $L^\infty$ norm bounded by $C\lambda$ for some $C > 0$.

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