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# A hypergeometric proof that Iso is bijective

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**Abstract**

We provide a short and elementary proof of the main technical result of the recent article "*On the uniqueness of Clifford torus with prescribed isoperimetric ratio*" [4] by Thomas Yu and Jingmin Chen. The key of the new proof is an explicit expression of the central function (Iso, to be proved bijective) as a quotient of Gaussian hypergeometric functions.

In their recent paper [4], Thomas Yu and Jingmin Chen needed to prove, as a crucial intermediate result, that a certain real-valued function Iso (related to isoperimetric ratios of Clifford tori) is strictly increasing. They reduced the proof of this fact to the positivity of a sequence of rational numbers $(d_n)_{n\geq 0}$, defined explicitly in terms of nested binomial sums. This positivity was subsequently proved by Stephen Melczer and Marc Mezzarobba [3], who used a computer-assisted approach relying on analytic combinatorics and rigorous numerics, combined with the fact (proved in [4]) that the sequence $(d_n)_{n\geq 0}$ satisfies an explicit linear recurrence of order seven with polynomial coefficients in $n$.

In this note, we provide an alternative, short and conceptual, proof of the monotonicity of the function Iso. Our approach is different in spirit from the ones in [4] and [3]. Our main result (Theorem 2 below) is that the function $\mathsf{Iso}(z)$ can be expressed in terms of Gaussian hypergeometric functions ${}_2F_1$ defined by

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the rising factorial $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$.

In the notation of Yu and Chen, the function

$$\mathsf{Iso} : [0, \sqrt{2}-1) \to [3/2 \cdot (2\pi^2)^{-1/4}, 1)$$

is given as

$$\mathsf{Iso}(z) = 6\sqrt{\pi} \cdot \frac{V(z)}{A^{3/2}(z)}, \tag{2}$$

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where $A(z) = \sum_{n\geq 0} a_n z^{2n}$ and $V(z) = \sum_{n\geq 0} v_n z^{2n}$ are complex analytic functions in the disk $\{z : |z| < \sqrt{2}-1\}$, given by the power series expansions

$$A(z) = \sqrt{2}\pi^2 \cdot \left(4 + 52z^2 + 477z^4 + 3809z^6 + \frac{451625}{16}z^8 + \cdots\right),$$

$$V(z) = \sqrt{2}\pi^2 \cdot \left(2 + 48z^2 + \frac{1269}{2}z^4 + 6600z^6 + \frac{1928025}{32}z^8 + \cdots\right).$$

The precise definitions of $A$ and $V$ are given in Section 4.3 of [4], notably in equations (4.2)–(4.3). Since the sequences $(a_n)_{\geq 0}$ and $(v_n)_{\geq 0}$ are expressed in terms of nested binomial sums, $A(z)$ and $V(z)$ satisfy linear differential equations with polynomial coefficients in $z$, that can be found and proved automatically using *creative telescoping* [2]. Yu and Chen, resp. Melczer and Mezzarobba, use this methodology to find a linear recurrence satisfied by the coefficients $(d_n)_{n\geq 0}$ of

$$F(z) := \frac{1}{4\pi^4}\cdot\left(\frac{2V'(\sqrt{z})A(\sqrt{z}) - 3V(\sqrt{z})A'(\sqrt{z})}{\sqrt{z}}\right) = 72+1932z+31248z^3+\cdots,$$

respectively a linear differential equation satisfied by the function $F(z)$.
Similarly, one can compute linear differential equations satisfied individually by

$$\bar{A}(z) := \frac{1}{\sqrt{2}\pi^2}\cdot A(\sqrt{z}) = 4+52\,z+477\,z^2+3809\,z^3+\frac{451625}{16}z^4+\frac{3195333}{16}z^5+\cdots$$

and by

$$\bar{V}(z) := \frac{1}{\sqrt{2}\pi^2}\cdot V(\sqrt{z}) = 2+48\,z+\frac{1269}{2}z^2+6600\,z^3+\frac{1928025}{32}z^4+\frac{2026101}{4}z^5+\cdots.$$

Concretely, $\bar{A}(z)$ and $\bar{V}(z)$ satisfy second-order linear differential equations:

$$\begin{aligned} z(z-1)(z^2-6z+1)(z+1)^2\bar{A}''(z)+(z+1)(5z^4-8z^3-32z^2+28z-1)\bar{A}'(z) \\ + \left(4z^4+11z^3-z^2-43z+13\right)\bar{A}(z) = 0 \end{aligned}$$

and respectively

$$\begin{aligned} z\,(z-1)\,(z+1)\left(z^2-6\,z+1\right)^2\bar{V}''(z) \\ +\left(z^2-6\,z+1\right)\left(7\,z^4-22\,z^3-18\,z^2+26\,z-1\right)\bar{V}'(z) \\ +3\,\left(3\,z^5-24\,z^4-2\,z^3+56\,z^2-25\,z+8\right)\bar{V}(z) = 0. \end{aligned}$$

From these equations, we deduce the following closed-form expressions:

**Theorem 1.** *The following equalities hold for all $z \in \mathbb{R}$ with $0 \leq z \leq \sqrt{2}-1$:*

$$\bar{A}(z) = \frac{4\left(1-z^2\right)}{\left(z^2-6z+1\right)^2}\cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2} \\ 1\end{matrix}; \frac{4z}{(1-z)^2}\right]$$

*and*

$$\bar{V}(z) = \frac{2\,(1-z)^3}{\left(z^2-6z+1\right)^3}\cdot {}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2} \\ 1\end{matrix}; \frac{4z}{(1-z)^2}\right].$$

*Proof.* It is enough to check that the right-hand side expressions satisfy the same linear differential equations as $\bar{A}$ and $\bar{V}$, with the same initial conditions. □

As a direct consequence of Theorem 1 and of definition (2) we get:

**Theorem 2.** *The function* $\mathsf{Iso}$ *admits the following closed-form expression:*

$$\mathsf{Iso}^2(z) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{{}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; \frac{4z^2}{(1-z^2)^2}\right]^2}{{}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; \frac{4z^2}{(1-z^2)^2}\right]^3} \cdot \left(\frac{1-z^2}{1+z^2}\right)^3.$$

Using the expression in Theorem 2, we can now prove the main result of [4].

**Theorem 3.** $\mathsf{Iso}$ *is a strictly increasing function and* $\lim_{z\to\sqrt{2}-1} \mathsf{Iso}(z) = 1$. *In particular,* $\mathsf{Iso}$ *is a bijection.*

*Proof.* The value of $\mathsf{Iso}^2(z)$ at $z = \sqrt{2}-1$ is equal to

$$\mathsf{Iso}^2(\sqrt{2}-1) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{{}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; 1\right]^2}{{}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; 1\right]^3} \cdot \frac{\sqrt{2}}{4}.$$

From Gauss's summation theorem [1, Th. 2.2.2] it follows that ${}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; 1\right] = 32/(3\pi)$ and ${}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; 1\right] = 4/\pi$; therefore,

$$\mathsf{Iso}^2(\sqrt{2}-1) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{(32/(3\pi))^2}{(4/\pi)^3} \cdot \frac{\sqrt{2}}{4} = 1.$$

It remains to prove that $\mathsf{Iso}$ is strictly increasing. It is enough to show that

$$z \;\mapsto\; \frac{{}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; \frac{4z}{(1-z)^2}\right]^2}{{}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; \frac{4z}{(1-z)^2}\right]^3} \cdot \left(\frac{1-z}{1+z}\right)^3$$

is increasing on $[0, 3-2\sqrt{2})$. Equivalently, via the change of variables $x = \frac{4z}{(1-z)^2}$, it is enough to prove that the function

$$h\colon x \;\mapsto\; \frac{{}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; x\right]^2}{{}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; x\right]^3} \cdot (x+1)^{-\frac{3}{2}}$$

is increasing on $[0, 1)$. Clearly, $h$ can be written as $h = f^3 \cdot g^2$, where

$$f(x) = \frac{\sqrt{x+1}}{{}_2F_1\left[\begin{smallmatrix}-\frac{1}{2} & -\frac{1}{2}\\ 1\end{smallmatrix}; x\right]} \quad \text{and} \quad g(x) = \frac{{}_2F_1\left[\begin{smallmatrix}-\frac{3}{2} & -\frac{3}{2}\\ 1\end{smallmatrix}; x\right]}{(x+1)^{\frac{3}{2}}}.$$

Hence, it is enough to prove that both $f$ and $g$ are increasing on $[0,1)$. We will actually prove a more general fact in Proposition 1, which may be of independent interest. Using that $w_{1/2} = 1/f$ and $w_{3/2} = g$, we deduce from Proposition 1 that both $f$ and $g$ are increasing. This concludes the proof of Theorem 3. □

**Proposition 1.** *Let $a \geq 0$ and let $w_a : [0,1] \to \mathbb{R}$ be defined by*

$$w_a(x) = \frac{{}_2F_1\left[\begin{smallmatrix} -a & -a \\ & 1 \end{smallmatrix}; x\right]}{(x+1)^a}.$$

*Then $w_a$ is: decreasing if $0 < a < 1$; increasing if $a > 1$; constant if $a \in \{0,1\}$.*

*Proof.* Clearly, if $a \in \{0,1\}$, then $w_a(x)$ is constant, equal to 1 on $[0,1]$.

Consider now the case $a > 0$ with $a \neq 1$. The derivative of $w_a(x)$ satisfies the hypergeometric identity

$$\frac{w_a'(x) \cdot (x+1)^{a+1}}{a \cdot (a-1) \cdot (1-x)^{2a}} = {}_2F_1\left[\begin{smallmatrix} a+1 & a \\ & 2 \end{smallmatrix}; x\right], \tag{3}$$

which is a direct consequence of Euler's transformation formula [1, Eq. (2.2.7), p. 68] and of Lemma 1 with $a$ substituted by $-a$.

Since $a > 0$, the right-hand side of (3) has only positive Taylor coefficients, therefore it is positive on $[0,1)$. It follows that $w_a'(x) \geq 0$ on $[0,1]$ if $a - 1 > 0$, and $w_a'(x) \leq 0$ on $[0,1]$ if $a - 1 < 0$. Equivalently, $w_a$ is increasing on $[0,1]$ if $a > 1$, and decreasing on $[0,1]$ if $a < 1$. □

**Lemma 1.** *The following identity holds:*

$$(a+1)(1-x) \cdot {}_2F_1\left[\begin{smallmatrix} a+1 & a+2 \\ & 2 \end{smallmatrix}; x\right] = a(x+1) \cdot {}_2F_1\left[\begin{smallmatrix} a+1 & a+1 \\ & 2 \end{smallmatrix}; x\right] + {}_2F_1\left[\begin{smallmatrix} a & a \\ & 1 \end{smallmatrix}; x\right].$$

*Proof.* We will use two of the classical Gauss' contiguous relations [1, §2.5]:

$${}_2F_1\left[\begin{smallmatrix} a+1 & b+1 \\ & c+1 \end{smallmatrix}; x\right] = \frac{c}{bx} \cdot \left({}_2F_1\left[\begin{smallmatrix} a+1 & b \\ & c \end{smallmatrix}; x\right] - {}_2F_1\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}; x\right]\right) \tag{4}$$

and

$$a \cdot \left({}_2F_1\left[\begin{smallmatrix} a+1 & b \\ & c \end{smallmatrix}; x\right] - {}_2F_1\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}; x\right]\right) = \frac{(c-b) \cdot {}_2F_1\left[\begin{smallmatrix} a & b-1 \\ & c \end{smallmatrix}; x\right] + (b - c + ax) \cdot {}_2F_1\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}; x\right]}{1-x}. \tag{5}$$

Applying (4) twice, once with $(b,c) = (a,1)$ and once with $(b,c) = (a+1,1)$, the proof of the lemma is reduced to that of the identity

$$(x-1) \cdot {}_2F_1\left[\begin{smallmatrix} a+1 & a+1 \\ & 1 \end{smallmatrix}; x\right] + 2 \cdot {}_2F_1\left[\begin{smallmatrix} a & a+1 \\ & 1 \end{smallmatrix}; x\right] = {}_2F_1\left[\begin{smallmatrix} a & a \\ & 1 \end{smallmatrix}; x\right],$$

which follows from (5) with $(b,c) = (a+1,1)$. □

**Remark 1.** A natural question is whether the function $\mathsf{lso}$ enjoys higher monotonicity properties. It can be easily seen that both $\mathsf{lso}$ and its reciprocal $1/\mathsf{lso}$ are neither convex nor concave. However we will prove that $z \mapsto \mathsf{lso}(\sqrt{z})$ is concave and $z \mapsto 1/\mathsf{lso}(\sqrt{z})$ is convex, on their domain of definition $[0, 3-2\sqrt{2})$.

First recall that $1/\mathsf{lso}(\sqrt{z}) = \frac{2^{5/4}\cdot\sqrt{\pi}}{3} \cdot w_{1/2}(r(z))^{3/2} \cdot w_{3/2}(r(z))^{-1}$, where we set $r(z) = 4z/(1-z)^2$. Since $w_{1/2} = 1/f$ and $w_{3/2}^{-1} = 1/g$ are positive and decreasing, while $r$ is nonnegative and increasing, proving that $w_{1/2} \circ r$ and $w_{3/2}^{-1} \circ r$ are both convex is enough to establish convexity of $z \mapsto 1/\mathsf{lso}(\sqrt{z})$.

From (3) and the chain rule it follows that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}z} w_a(r(z))}{4\,a\cdot(a-1)} = {}_2F_1\left[\begin{matrix} a+1 & a \\ 2 \end{matrix}; \frac{4z}{(1-z)^2}\right] \cdot \frac{(1-6\,z+z^2)^{2a}}{(1-z)^{4a}} \cdot \frac{(1-z)^{2a-1}}{(1+z)^{2a+1}}. \tag{6}$$

We can justify convexity of both $w_{1/2}(r(z))$ and $w_{3/2}(r(z))^{-1}$ if we can prove that the right-hand side of (6) is decreasing on $[0, 3-2\sqrt{2})$. Moreover, it is easy to see that $(1-z)^{2a-1}/(1+z)^{2a+1}$ is decreasing on this interval for $a > 3/2-\sqrt{2}$. Therefore, after changing variables $x = 4z/(1-z)^2$, it remains to show that

$${}_2F_1\left[\begin{matrix} a+1 & a \\ 2 \end{matrix}; \frac{4z}{(1-z)^2}\right] \cdot \frac{(1-6\,z+z^2)^{2a}}{(1-z)^{4a}} = {}_2F_1\left[\begin{matrix} a+1 & a \\ 2 \end{matrix}; x\right] \cdot (1-x)^{2a}$$

is decreasing for all $x \in [0,1)$. The derivative of the right-hand side is given by

$$-\left(\frac{a(3-a)}{2} \cdot {}_2F_1\left[\begin{matrix} a & a+1 \\ 3 \end{matrix}; x\right] + \frac{a(a+1)x}{6} \cdot {}_2F_1\left[\begin{matrix} a+1 & a+2 \\ 4 \end{matrix}; x\right]\right) \cdot (1-x)^{2a-1},$$

hence is indeed negative for all $x \in [0,1)$ if $0 < a < 3$. From this and (6) it follows that $1/\mathsf{lso}(\sqrt{z})$ is the product of two positive, decreasing and convex functions and therefore inherits these properties. Finally, this also shows that $\mathsf{lso}(\sqrt{z})$ is both increasing and concave on $[0, 3-2\sqrt{2})$.

**Remark 2.** Bruno Salvy (private communication) found an alternative short proof of Proposition 1. The main idea is inspired by the Sturm–Liouville theory and the proof is based on the observation that $w_a(x)$ satisfies the linear differential equation (written in adjoint form):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(\frac{1+x}{1-x}\right)^{2a} \cdot \frac{\mathrm{d}}{\mathrm{d}x} w_a(x)\right) = \frac{a(a-1)x}{(1+x)^2}\left(\frac{1+x}{1-x}\right)^{4a} \cdot w_a(x).$$

The right-hand side is positive on $(0,1)$ when $a > 1$ and negative if $0 < a < 1$. The same holds for its integral over $[0,t]$ for any $t < 1$. Looking at the left-hand side, this implies that $w_a' > 0$ whenever $a > 1$ and $w_a' < 0$ when $0 < a < 1$.

We note that the same idea allows for a different proof of Remark 1.

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