



Transient Performance Analysis of the L1-RLS

Wei Gao, Jie Chen, Cédric Richard, Wentao Shi, Qunfei Zhang

► To cite this version:

Wei Gao, Jie Chen, Cédric Richard, Wentao Shi, Qunfei Zhang. Transient Performance Analysis of the L1-RLS. IEEE Signal Processing Letters, 2021, 29, pp.90-94. 10.1109/LSP.2021.3127467 . hal-03347324

HAL Id: hal-03347324

<https://hal.science/hal-03347324>

Submitted on 17 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Transient Performance Analysis of the ℓ_1 -RLS

Wei Gao, *Member, IEEE*, Jie Chen, *Senior Member, IEEE*, Cédric Richard *Senior Member, IEEE*,
Wentao Shi, *Member, IEEE*, and Qunfei Zhang, *Member, IEEE*

Abstract—The recursive least-squares algorithm with ℓ_1 -norm regularization (ℓ_1 -RLS) exhibits excellent performance in terms of convergence rate and steady-state error in identification of sparse systems. Nevertheless few works have studied its stochastic behavior, in particular its transient performance. In this letter, we derive analytical models of the transient behavior of the ℓ_1 -RLS in the mean and mean-square sense. Simulation results illustrate the accuracy of these models.

Index Terms—Transient analysis, sparse system, online identification, ℓ_1 -RLS.

I. INTRODUCTION

SPARSITY aware adaptive filters have been successfully applied to a wide range of applications, e.g., echo cancellation [1], channel estimation [2], and system identification [3]. As the recursive least-squares (RLS) algorithm achieves better performance than the least-mean-square (LMS) algorithm for time-invariant system identification [4], [5], sparse RLS-type algorithms have attracted considerable attention.

The recursive ℓ_1 -regularized least-squares (SPARLS) algorithm was introduced using the expectation-maximization scheme [6]. The greedy sparse RLS algorithm with exponential window exploiting the orthogonal matching pursuit was devised in [7]. The ℓ_1 -norm regularized RLS (ℓ_1 -RLS) algorithm was proposed based on the modified least-squares cost function with sparsity promoting regularization [8]. By considering two possible weighted ℓ_1 -norm sparsity constraints with the cost function, two weighted ℓ_1 -RLS algorithms were presented in [9]. They can be viewed as particular cases of the convex regularized RLS (CR-RLS) algorithm, that was derived by considering any convex function in the regularizer [10]. In [11], the cost function of the original RLS was modified by adding an adaptively weighted ℓ_2 -norm penalty resulting in the proposal of two zero-attracting RLS (ZA-RLS) algorithms. A distributed sparse RLS algorithm was also proposed for decentralized scenarios over networks in [12]–[14].

The zero-attracting LMS (ZA-LMS) was extensively studied in [15]–[17], with a focus on its transient behavior. In [18], the authors analyzed the mean and mean-square deviations at steady-state of the ℓ_0 -norm regularized RLS (ℓ_0 -RLS) algorithm. In contrast, despite the superiority of the ℓ_1 -RLS

algorithm, no theoretical analysis of its transient behavior has been reported so far. This may be due to the fact that there are few analyses of the transient behavior of the RLS algorithm in the literature. In addition, the analysis of the update equation in the ℓ_1 -RLS is critical from a statistical perspective. To address this concern, we start this letter by reformulating the update equation of the ℓ_1 -RLS algorithm into a concise form. This makes the analysis of its transient behavior tractable both in the mean and mean-square sense. Simulation results validate the theoretical findings.

Notation: $[\mathbf{x}]_i$ and $[\mathbf{X}]_{ij}$ denote the i -th entry of column vector \mathbf{x} and the (i, j) -th entry of matrix \mathbf{X} , respectively. The superscript $(\cdot)^\top$ represents the transpose of vector or matrix. The matrix trace is denoted by $\text{tr}\{\cdot\}$. The operator $\text{sgn}\{\cdot\}$ takes the sign of the entries of its argument. All-zero vector of length N is denoted by $\mathbf{0}_N$, and all-one vector of length N is denoted by $\mathbf{1}_N$. The Gaussian distribution with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$. The multivariate Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is denoted by $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The cumulative distribution function (CDF) of the standard Gaussian distribution is denoted by $\phi(x)$. The CDF of the multivariate Gaussian distribution is denoted by $\Phi(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$.

II. THE PROBLEM AND ℓ_1 -RLS ALGORITHM

Assume that the input-output sequences are generated by an unknown time-invariant system with sparse impulse response:

$$y_n = \mathbf{x}_n^\top \mathbf{w}^* + z_n \quad (1)$$

where $\mathbf{x}_n \in \mathbb{R}^L$ is the regression vector at time instant n with positive definite correlation matrix $\mathbf{R}_x = \mathbb{E}\{\mathbf{x}_n \mathbf{x}_n^\top\}$, and $\mathbf{w}^* \in \mathbb{R}^L$ is the sparse optimal weight vector to be estimated. The modeling error z_n is assumed to be stationary, white and Gaussian with zero-mean and variance σ_z^2 , and statistically independent of any other signal. Consider the batch least-absolute shrinkage and selection operator (LASSO) estimator usually considered for sparse system identification problems [19]:

$$\mathbf{w}^o = \arg \min_{\mathbf{w} \in \mathbb{R}^L} \left\{ \frac{1}{2} \sum_{i=0}^n \lambda^{n-i} |y_i - \mathbf{w}^\top \mathbf{x}_i|^2 + \delta \|\mathbf{w}\|_1 \right\} \quad (2)$$

with $0 \ll \lambda < 1$ a forgetting factor, and $\delta > 0$ a regularization parameter that controls the trade-off between the estimation error and the sparsity of the weight vector.

Based on the modified deterministic normal equation resulting from the subgradient vector of (2), the ℓ_1 -RLS algorithm

Manuscript received MM DD, 2021; revised MM DD, 2021.

This work was supported in part by the National NSFC under Grants 62171205 and 62171380.

W. Gao is with the School of Computer Science and Telecommunication Engineering, Jiangsu University, Zhenjiang 212013, China (email: wei_gao@ujs.edu.cn).

J. Chen, W. Shi, and Q. Zhang are with the School of Marine Science and Technology, Northwestern Polytechnical University, Xi'an 710072, China (email: dr.jie.chen@ieee.org; swt@nwpu.edu.cn; zhangqf@nwpu.edu.cn).

C. Richard is with the Université Côte d'Azur, CNRS, OCA, 06108 Nice, France (e-mail: cedric.richard@unice.fr).

proposed in [8]–[10] is given by:

$$\mathbf{k}_n = \frac{\mathbf{P}_{n-1}\mathbf{x}_n}{\lambda + \mathbf{x}_n^\top \mathbf{P}_{n-1} \mathbf{x}_n}, \quad (3)$$

$$\mathbf{w}_n = \mathbf{w}_{n-1} + e_n \mathbf{k}_n + \delta \left(\frac{\lambda - 1}{\lambda} \right) (\mathbf{I}_L - \mathbf{k}_n \mathbf{x}_n^\top) \times \mathbf{P}_{n-1} \text{sgn}\{\mathbf{w}_{n-1}\}, \quad (4)$$

$$\mathbf{P}_n = \lambda^{-1}(\mathbf{P}_{n-1} - \mathbf{k}_n \mathbf{x}_n^\top \mathbf{P}_{n-1}), \quad (5)$$

with the instantaneous estimation error $e_n = y_n - \mathbf{x}_n^\top \mathbf{w}_{n-1}$. Here, matrix \mathbf{P}_n is defined as the inverse of the time-averaged correlation matrix Φ_n for the input vector [4], [5], given by:

$$\Phi_n = \sum_{i=0}^n \lambda^{n-i} \mathbf{x}_i \mathbf{x}_i^\top + \delta \lambda^{n+1} \mathbf{I}_L = \lambda \Phi_{n-1} + \mathbf{x}_n \mathbf{x}_n^\top. \quad (6)$$

Recursion (4) is too complex to use in convergence analysis. By inserting $\mathbf{k}_n = \mathbf{P}_n \mathbf{x}_n$ and $\mathbf{I}_L - \mathbf{k}_n \mathbf{x}_n^\top = \lambda \mathbf{P}_n \mathbf{P}_{n-1}^{-1}$ from (3) and (5), respectively, into (4), it can be simplified as follows:

$$\mathbf{w}_n = \mathbf{w}_{n-1} + e_n \mathbf{P}_n \mathbf{x}_n + \gamma \mathbf{P}_n \text{sgn}\{\mathbf{w}_{n-1}\} \quad (7)$$

with $\gamma = \delta(\lambda - 1)$. This alternative recursion of the ℓ_1 -RLS algorithm makes the following analysis easier to handle. It can be seen from (7) that the only difference between the RLS and the ℓ_1 -RLS lies in the rightmost zero attractor term.

III. TRANSIENT PERFORMANCE ANALYSIS OF ℓ_1 -RLS

We shall now study the transient behavior of the ℓ_1 -RLS algorithm in the mean and mean-square sense. We define the weight error vector $\tilde{\mathbf{w}}_n$ as the difference between the weight vector \mathbf{w}_n and \mathbf{w}^* , i.e.,

$$\tilde{\mathbf{w}}_n = \mathbf{w}_n - \mathbf{w}^*. \quad (8)$$

In essence, the analysis of the ℓ_1 -RLS consists of studying the evolution over time of the expectation of $\tilde{\mathbf{w}}_n$ and its correlation matrix $\mathbf{K}_n = \mathbb{E}\{\tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top\}$.

Before proceeding, we introduce the following statistical assumptions for mathematical tractability.

A1: The weight error vector $\tilde{\mathbf{w}}_{n-1}$ is statistically independent of the regression vector \mathbf{x}_n .

A2: Any pair of entries $[\tilde{\mathbf{w}}_n]_i$ and $[\tilde{\mathbf{w}}_n]_j$ with $i \neq j$ is jointly Gaussian.

The so-called independence assumption (IA) A1 is widely used in the convergence analysis of adaptive filters [4], [5]. Assumption A2 has been used successfully in the analysis of the ZA-LMS algorithm [17] as it makes the analysis of the nonlinear sign term in (7) tractable. We shall illustrate the validity of A2 with histograms of $[\tilde{\mathbf{w}}_n]_i$ versus $[\tilde{\mathbf{w}}_n]_j$.

A. Mean Weight Error Behavior Model

We focus on the mean weight error analysis of the ℓ_1 -RLS. Taking the expectation of both sides of (6) yields:

$$\mathbb{E}\{\Phi_n\} = \lambda \mathbb{E}\{\Phi_{n-1}\} + \mathbf{R}_x \quad (9)$$

with $\Phi_0 = \varepsilon^{-1} \mathbf{I}_L$ and ε a positive number. This recursion will be used in the following analysis. From (1) and (8), the instantaneous estimation error e_n can be rewritten as follows:

$$e_n = z_n - \mathbf{x}_n^\top \tilde{\mathbf{w}}_{n-1}. \quad (10)$$

Subtracting \mathbf{w}^* from both sides of (7), using (8) and (10), we have:

$$\tilde{\mathbf{w}}_n = \tilde{\mathbf{w}}_{n-1} - \mathbf{P}_n \mathbf{x}_n \mathbf{x}_n^\top \tilde{\mathbf{w}}_{n-1} + z_n \mathbf{P}_n \mathbf{x}_n + \gamma \mathbf{P}_n \text{sgn}\{\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}\}. \quad (11)$$

Pre-multiplying both sides of (11) by \mathbf{P}_n^{-1} , then applying (6) and definition $\Phi_n = \mathbf{P}_n^{-1}$, leads to

$$\Phi_n \tilde{\mathbf{w}}_n = \lambda \Phi_{n-1} \tilde{\mathbf{w}}_{n-1} + z_n \mathbf{x}_n + \gamma \text{sgn}\{\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}\}. \quad (12)$$

Taking the expectation of both sides of (12), considering that noise z_n is statistically independent of any other signal and it is zero mean, we have:

$$\mathbb{E}\{\Phi_n \tilde{\mathbf{w}}_n\} = \lambda \mathbb{E}\{\Phi_{n-1} \tilde{\mathbf{w}}_{n-1}\} + \gamma \mathbb{E}\{\text{sgn}\{\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}\}\}. \quad (13)$$

Using the following approximation presented in detail in [20], and valid for small values of λ :

$$\mathbb{E}\{\Phi_n \tilde{\mathbf{w}}_n\} \approx \mathbb{E}\{\Phi_n\} \mathbb{E}\{\tilde{\mathbf{w}}_n\} \quad (14)$$

equation (13) becomes:

$$\mathbb{E}\{\Phi_n\} \mathbb{E}\{\tilde{\mathbf{w}}_n\} = \lambda \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\tilde{\mathbf{w}}_{n-1}\} + \gamma \mathbb{E}\{\text{sgn}\{\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}\}\}. \quad (15)$$

Let $u_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ a Gaussian random variable. Lemma 1 presented in [17] shows that:

$$\mathbb{E}\{\text{sgn}\{u_i\}\} = 1 - 2\phi(-\mu_i/\sigma_i). \quad (16)$$

In order to evaluate the last term on the r.h.s. of (15), we set the i -th entry of $\mathbb{E}\{\text{sgn}\{\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}\}\}$ as follows:

$$[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_i \rightarrow u_i \quad (17)$$

with

$$[\mathbf{w}^*]_i + \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i\} \rightarrow \mu_i, \quad (18)$$

$$\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i^2\} - \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i\}^2 \rightarrow \sigma_i^2, \quad (19)$$

where $\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i^2\}$ can be extracted from the main diagonal entries of matrix \mathbf{K}_{n-1} that will be determined in the next subsection. With the above definitions, it then follows that:

$$\mathbb{E}\{\Phi_n\} \mathbb{E}\{\tilde{\mathbf{w}}_n\} = \lambda \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\tilde{\mathbf{w}}_{n-1}\} + \gamma (\mathbf{I}_L - 2\boldsymbol{\varphi}) \quad (20)$$

with $\boldsymbol{\varphi} = [\phi(-\mu_1/\sigma_1), \phi(-\mu_2/\sigma_2), \dots, \phi(-\mu_L/\sigma_L)]^\top$. By pre-multiplying both sides of (20) by $\mathbb{E}\{\Phi_n\}^{-1}$, we arrive at the mean weight error behavior for the ℓ_1 -RLS algorithm:

$$\mathbb{E}\{\tilde{\mathbf{w}}_n\} = \lambda \mathbb{E}\{\Phi_n\}^{-1} \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\tilde{\mathbf{w}}_{n-1}\} + \gamma \mathbb{E}\{\Phi_n\}^{-1} (\mathbf{I}_L - 2\boldsymbol{\varphi}) \quad (21)$$

where (9) is used to perform the above recursion.

B. Mean-Square Error Behavior Model

We shall now analyze the ℓ_1 -RLS algorithm in the mean-square sense. Squaring (10) and taking its expected value, then

considering assumption A1 and the statistical property of z_n , the mean-square error (MSE) can be expressed as

$$\mathbb{E}\{e_n^2\} \approx \sigma_z^2 + \text{tr}\{\mathbf{R}_x \mathbf{K}_{n-1}\} \quad (22)$$

where the second term on the r.h.s. of the above equation denotes the excess mean-square error (EMSE) [4], [5]. Moreover, the instantaneous mean-square-deviation (MSD) is defined by:

$$\text{MSD}_n = \mathbb{E}\{\|\tilde{\mathbf{w}}_n\|^2\} = \text{tr}\{\mathbf{K}_n\}. \quad (23)$$

In order to evaluate the EMSE or the MSE, and the MSD, we have to determine a recursion to calculate the correlation matrix \mathbf{K}_n . Post-multiplying (12) by its transpose, taking the expectation of both sides, and using the statistical property of noise z_n , yields

$$\begin{aligned} \mathbb{E}\{\Phi_n \tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top \Phi_n\} &= \lambda^2 \mathbb{E}\{\Phi_{n-1} \tilde{\mathbf{w}}_{n-1} \tilde{\mathbf{w}}_{n-1}^\top \Phi_{n-1}\} + \sigma_z^2 \mathbf{R}_x \\ &+ \gamma^2 \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\} \\ &+ \lambda\gamma \mathbb{E}\{\Phi_{n-1} \tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\} \\ &+ \lambda\gamma \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top \Phi_{n-1}\}. \end{aligned} \quad (24)$$

Evaluating (24) is difficult without using some approximations. The following approximations are used in the sequel:

$$\mathbb{E}\{\Phi_n \tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top \Phi_n\} \approx \mathbb{E}\{\Phi_n\} \mathbf{K}_n \mathbb{E}\{\Phi_n\}, \quad (25)$$

$$\begin{aligned} \mathbb{E}\{\Phi_{n-1} \tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\} \\ \approx \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\}, \end{aligned} \quad (26)$$

$$\begin{aligned} \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top \Phi_{n-1}\} \\ \approx \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top\} \mathbb{E}\{\Phi_{n-1}\}. \end{aligned} \quad (27)$$

Similar approximations are presented in detail in [20]. To understand these approximations, we denote by Δ_n the random fluctuation of matrix Φ_n around $\mathbb{E}\{\Phi_n\}$, namely,

$$\Phi_n = \mathbb{E}\{\Phi_n\} + \Delta_n. \quad (28)$$

Then, we have:

$$\begin{aligned} \mathbb{E}\{\Phi_n \tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top \Phi_n\} &= \mathbb{E}\{\Phi_n\} \mathbf{K}_n \mathbb{E}\{\Phi_n\} \\ &+ \mathbb{E}\{\Phi_n\} \mathbb{E}\{\tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top \Delta_n\} \\ &+ \mathbb{E}\{\Delta_n \tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top\} \mathbb{E}\{\Phi_n\} \\ &+ \mathbb{E}\{\Delta_n \tilde{\mathbf{w}}_n \tilde{\mathbf{w}}_n^\top \Delta_n\}, \end{aligned} \quad (29)$$

$$\begin{aligned} \mathbb{E}\{\Phi_{n-1} \tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\} \\ = \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\} \\ + \mathbb{E}\{\Delta_{n-1} \tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\}, \end{aligned} \quad (30)$$

$$\begin{aligned} \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top \Phi_{n-1}\} \\ = \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top\} \mathbb{E}\{\Phi_{n-1}\} \\ + \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \tilde{\mathbf{w}}_{n-1}^\top \Delta_{n-1}\}. \end{aligned} \quad (31)$$

We assume that the entries of Δ_n are small with respect to those of $\mathbb{E}\{\Phi_n\}$ based on the fact that (6) shows that Φ_n is a low-pass filtering of $\mathbf{x}_n \mathbf{x}_n^\top$. The first term on the r.h.s. of each one of the above three equations then dominates the remaining ones, which leads to (25)–(27). Simulation results in the sequel will confirm the validity of these approximations.

Using (25)–(27), expression (24) can then be written as:

$$\begin{aligned} \mathbb{E}\{\Phi_n\} \mathbf{K}_n \mathbb{E}\{\Phi_n\} &= \lambda^2 \mathbb{E}\{\Phi_{n-1}\} \mathbf{K}_{n-1} \mathbb{E}\{\Phi_{n-1}\} + \sigma_z^2 \mathbf{R}_x \\ &+ \gamma^2 \mathbf{Q}_1 + \lambda\gamma [\mathbb{E}\{\Phi_{n-1}\} \mathbf{Q}_2 + \mathbf{Q}_2^\top \mathbb{E}\{\Phi_{n-1}\}] \end{aligned} \quad (32)$$

with

$$\mathbf{Q}_1 = \mathbb{E}\{\text{sgn}\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\}, \quad (33)$$

$$\mathbf{Q}_2 = \mathbb{E}\{\tilde{\mathbf{w}}_{n-1} \text{sgn}^\top\{\mathbf{w}_n^* + \tilde{\mathbf{w}}_{n-1}\}\}. \quad (34)$$

Pre-multiplying and post-multiplying (32) by $\mathbb{E}\{\Phi_n\}^{-1}$ simultaneously, it results that:

$$\begin{aligned} \mathbf{K}_n &= \lambda^2 \mathbb{E}\{\Phi_n\}^{-1} \mathbb{E}\{\Phi_{n-1}\} \mathbf{K}_{n-1} \mathbb{E}\{\Phi_{n-1}\} \mathbb{E}\{\Phi_n\}^{-1} \\ &+ \sigma_z^2 \mathbb{E}\{\Phi_n\}^{-1} \mathbf{R}_x \mathbb{E}\{\Phi_n\}^{-1} + \gamma^2 \mathbb{E}\{\Phi_n\}^{-1} \mathbf{Q}_1 \mathbb{E}\{\Phi_n\}^{-1} \\ &+ \lambda\gamma \mathbb{E}\{\Phi_n\}^{-1} [\mathbb{E}\{\Phi_{n-1}\} \mathbf{Q}_2 + \mathbf{Q}_2^\top \mathbb{E}\{\Phi_{n-1}\}] \mathbb{E}\{\Phi_n\}^{-1}. \end{aligned} \quad (35)$$

In order to perform recursion (35), we shall now calculate \mathbf{Q}_1 and \mathbf{Q}_2 . Before proceeding by using assumption A2, consider two jointly Gaussian random variables u and v defined by:

$$\begin{bmatrix} u \\ v \end{bmatrix} \sim \mathcal{N}\left(\boldsymbol{\mu} := \begin{bmatrix} \mu_u \\ \mu_v \end{bmatrix}, \boldsymbol{\Sigma}_{uv} := \begin{bmatrix} \sigma_u^2 & \rho_{uv} \\ \rho_{uv} & \sigma_v^2 \end{bmatrix}\right) \quad (36)$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_{uv}$ denote the mean vector and the covariance matrix, respectively. According to Lemma 2 presented in [17], it holds that:

$$\begin{aligned} \mathbb{E}\{\text{sgn}\{u\} \text{sgn}\{v\}\} &= \Phi(\mathbf{0}_2, [\mu_u, \mu_v]^\top, \boldsymbol{\Sigma}_{uv}) \\ &+ \Phi(\mathbf{0}_2, -[\mu_u, \mu_v]^\top, \boldsymbol{\Sigma}_{uv}) - \Phi(\mathbf{0}_2, [\mu_u, -\mu_v]^\top, \bar{\boldsymbol{\Sigma}}_{uv}) \\ &- \Phi(\mathbf{0}_2, [-\mu_u, \mu_v]^\top, \bar{\boldsymbol{\Sigma}}_{uv}) \end{aligned} \quad (37)$$

with

$$\bar{\boldsymbol{\Sigma}}_{uv} = \begin{bmatrix} \sigma_u^2 & -\rho_{uv} \\ -\rho_{uv} & \sigma_v^2 \end{bmatrix}. \quad (38)$$

On the one hand, observe that the main diagonal entries $[\mathbf{Q}_1]_{ii}$ for $1 \leq i \leq L$ are equal to 1. On the other hand, considering assumption A2 and (37), the off-diagonal entries $[\mathbf{Q}_1]_{ij}$ for $1 \leq i \neq j \leq L$ can be obtained by making the identifications:

$$[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_i \rightarrow u, \quad (39)$$

$$[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_j \rightarrow v, \quad (40)$$

with

$$\mathbb{E}\{[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_i\} \rightarrow \mu_u, \quad (41)$$

$$\mathbb{E}\{[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_j\} \rightarrow \mu_v, \quad (42)$$

$$\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i^2\} - \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i\}^2 \rightarrow \sigma_u^2, \quad (43)$$

$$\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_j^2\} - \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_j\}^2 \rightarrow \sigma_v^2, \quad (44)$$

$$\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i [\tilde{\mathbf{w}}_{n-1}]_j\} - \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i\} \mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_j\} \rightarrow \rho_{uv}, \quad (45)$$

where the expectations $\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i [\tilde{\mathbf{w}}_{n-1}]_j\}$ are available from the off-diagonal entries $[\mathbf{K}_{n-1}]_{ij}$. Furthermore, based on definition (36) and Lemma 3 proved in [17], it holds that:

$$\mathbb{E}\{u \text{sgn}\{v\}\} \quad (46)$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi a |\boldsymbol{\Sigma}_{uv}|}} \left\{ \sqrt{\frac{2\pi}{\theta}} \left(\mu_u + \frac{c}{a} \mu_v \right) \left[1 - 2\phi(-\mu_v \sqrt{\theta}) \right] \right. \\ &\quad \left. - \frac{c}{a} \sqrt{\frac{2\pi}{\theta}} \left[\sqrt{\frac{2}{\pi\theta}} \exp\left(-\frac{1}{2}\mu_v^2 \theta\right) + \mu_v \left(1 - 2\phi(-\mu_v \sqrt{\theta}) \right) \right] \right\} \end{aligned}$$

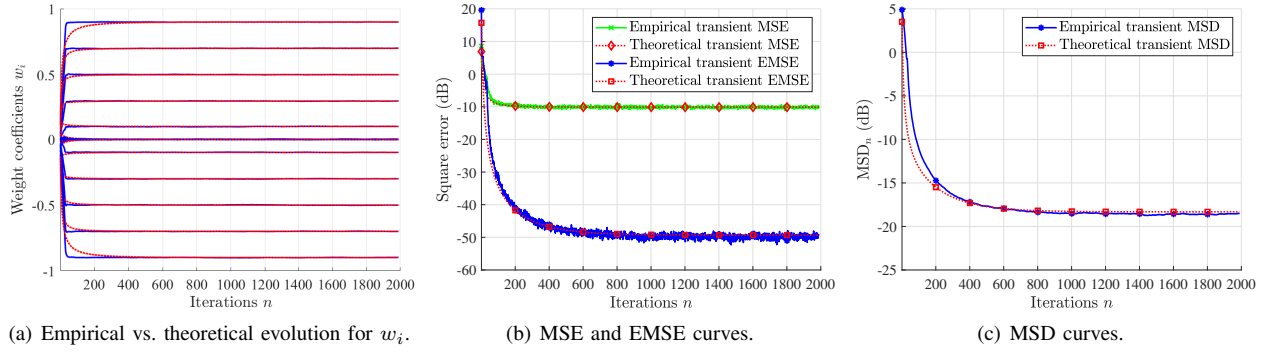


Fig. 2. Comparisons of empirical and theoretical simulation results for the ℓ_1 -RLS algorithm.

where $\theta = b - c^2/a > 0$ with

$$\Sigma_{uv}^{-1} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}. \quad (47)$$

Likewise, in order to evaluate matrix \mathbf{Q}_2 , we can make the following identifications:

$$[\tilde{\mathbf{w}}_{n-1}]_i \rightarrow u, \quad (48)$$

$$[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_j \rightarrow v, \quad (49)$$

with

$$\mathbb{E}\{[\tilde{\mathbf{w}}_{n-1}]_i\} \rightarrow \mu_u, \quad (50)$$

$$\mathbb{E}\{[\mathbf{w}^* + \tilde{\mathbf{w}}_{n-1}]_j\} \rightarrow \mu_v, \quad (51)$$

for $1 \leq i, j \leq L$. Consequently, all the entries of matrix \mathbf{Q}_2 can be determined according to (46) under assumption A2. Using (9), (33), and (34), we can finally perform recursion (35), which allows us to characterize the transient mean-square error behavior of the ℓ_1 -RLS algorithm.

IV. NUMERICAL TESTS

The good performance of the ℓ_1 -RLS algorithm has already been illustrated in the literature by comparing it with related adaptive filters for sparse system identification [8]–[10]. We shall now examine the accuracy of the analytical models derived in this paper, via simulation results. All empirical curves were obtained by averaging over 500 Monte Carlo runs.

The input signal was generated with a first-order AR model, namely, $x_n = 0.6x_{n-1} + s_n$, with s_n a zero-mean white Gaussian random sequence. Its variance was set to $\sigma_s^2 = 0.64$, in order that the variance of the input signal x_n was $\sigma_x^2 = 1$. The noise z_n was zero-mean white and Gaussian with variance $\sigma_z^2 = 0.09$. The optimal weight vector in (1) was set to:

$$\mathbf{w}^* = [0.9, 0.7, 0.5, 0.3, 0.1, \mathbf{0}_{22}^\top, -0.1, -0.3, -0.5, -0.7, -0.9]^\top \in \mathbb{R}^{32}. \quad (52)$$

The exponential forgetting factor was set to $\lambda = 0.995$, and the regularization parameter was set to $\delta = 0.25$. The initial parameter ε was set to 0.1. The weight vector was initialized to zero, that is, $\mathbf{w}_0 = \mathbf{0}_{32}$.

Four histograms are depicted in Fig. 1 for two arbitrarily selected pairs of entries of the weight error vector $\tilde{\mathbf{w}}_n$, i.e., $[\tilde{\mathbf{w}}_n]_2, [\tilde{\mathbf{w}}_n]_{10}]$ and $[\tilde{\mathbf{w}}_n]_{13}, [\tilde{\mathbf{w}}_n]_{25}]$ from 5×10^3 simulated samples, at time instant $n = 200$ and $n = 1500$, respectively. As shown in Fig. 1, the four histograms of bivariate vector

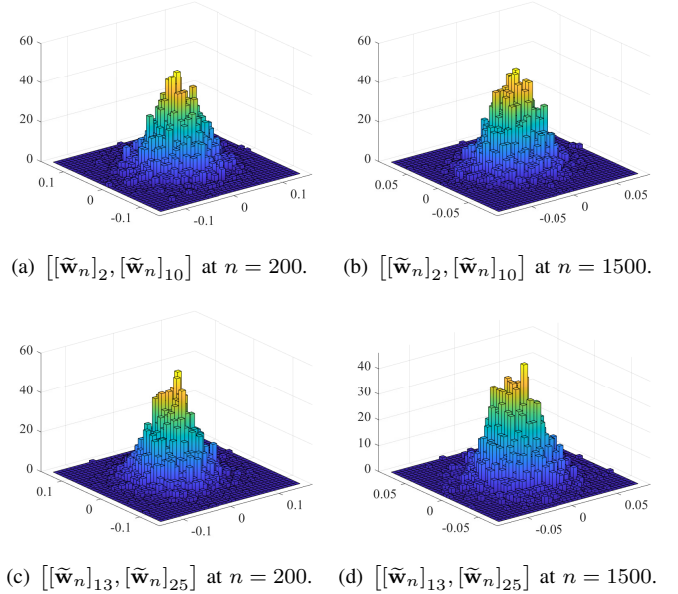


Fig. 1. Histograms of bivariate vector $[\tilde{\mathbf{w}}_n]_i, [\tilde{\mathbf{w}}_n]_j$ with 5×10^3 samples.

$[\tilde{\mathbf{w}}_n]_i, [\tilde{\mathbf{w}}_n]_j$ have the required Gaussian-like profiles, which validates the feasibility of assumption A2. The mean weight behavior is shown in Fig. 2(a). One can observe that all the theoretical curves of weight coefficients w_i predicted by (21) are generally consistent with those empirical curves including the zero coefficients. Fig. 2(b) shows that the empirical learning curves of the MSE and EMSE coincide with their theoretical curves obtained from (22) and (35), respectively. Fig. 2(c) shows the good agreement between the empirical learning curve of MSD and its theoretical prediction obtained from (23) and (35). Notice that there is a mismatch between the simulated and theoretical results during the initial transient stage due to the approximations (25)–(27). Fig. 2 illustrates the correctness and accuracy of our analytical models and the necessary approximations used in the analysis. These models offer an effective means of profoundly understanding the convergence behavior of the ℓ_1 -RLS algorithm in the context of online sparse system identification.

V. CONCLUSION

In this letter, the transient behavior of the ℓ_1 -RLS algorithm was theoretically studied in the mean and mean-square sense. Simulation results illustrated their accuracy.

REFERENCES

- [1] C. Elisei-Iliescu, C. Paleologu, J. Benesty, C. Stanciu, C. Anghel, and S. Ciochină, "Recursive least-squares algorithms for the identification of low-rank systems," *IEEE/ACM Trans. Audio, Speech, Language Process.*, vol. 27, no. 5, pp. 903–918, May 2019.
- [2] K. Pelekanakis and M. Chitre, "Adaptive sparse channel estimation under symmetric alpha-stable noise," *IEEE Trans. Wireless Commun.*, vol. 13, no. 6, pp. 3183–3195, Jun. 2014.
- [3] Y. V. Zakharov and V. H. Nascimento, "DCD-RLS adaptive filters with penalties for sparse identification," *IEEE Trans. on Signal Process.*, vol. 61, no. 12, pp. 3198–3213, Jun. 2013.
- [4] S. Haykin, *Adaptive Filter Theory*, 2nd ed. New Jersey: Prentice-Hall, 1991.
- [5] A. H. Sayed, *Fundamentals of Adaptive Filtering*. New York: Wiley, 2003.
- [6] B. Babadi, N. Kalouptsidis, and V. Tarokh, "SPARLS: The sparse RLS algorithm," *IEEE Trans. on Signal Process.*, vol. 58, no. 8, pp. 4013–4025, Aug. 2010.
- [7] B. Dumitrescu, A. Onose, P. Helin, and I. Tabus, "Greedy sparse RLS," *IEEE Trans. on Signal Process.*, vol. 60, no. 5, pp. 2194–2207, May 2012.
- [8] E. M. Eksioğlu, "RLS adaptive filtering with sparsity regularization," in *10th International Conference on Information Science, Signal Processing and their Applications (ISSPA 2010)*, 2010, pp. 550–553.
- [9] E. M. Eksioğlu, "Sparsity regularised recursive least squares adaptive filtering," *IET Signal Processing*, vol. 5, pp. 480–487, Aug. 2011.
- [10] E. M. Eksioğlu and A. K. Tanc, "RLS algorithm with convex regularization," *IEEE Signal Process. Lett.*, vol. 18, no. 8, pp. 470–473, Aug. 2011.
- [11] X. Hong, J. Gao, and S. Chen, "Zero-attracting recursive least squares algorithms," *IEEE Trans. Veh. Technol.*, vol. 66, no. 1, pp. 213–221, Jan. 2017.
- [12] Z. Liu, Y. Liu, and C. Li, "Distributed sparse recursive least-squares over networks," *IEEE Trans. on Signal Process.*, vol. 62, no. 6, pp. 1386–1395, Mar. 2014.
- [13] X. Cao and K. J. R. Liu, "Decentralized sparse multitask RLS over networks," *IEEE Trans. on Signal Process.*, vol. 65, no. 23, pp. 6217–6232, Dec. 2017.
- [14] P. M. Djuric and C. Richard, *Cooperative and Graph Signal Processing: Principles and Applications*. New York, USA: Academic, 2018.
- [15] K. Shi and P. Shi, "Convergence analysis of sparse LMS algorithms with l_1 -norm penalty based on white input signal," *Signal Process.*, vol. 90, no. 12, pp. 3289–3293, Dec. 2010.
- [16] S. Zhang and J. Zhang, "Transient analysis of zero attracting NLMS algorithm without Gaussian inputs assumption," *Signal Process.*, vol. 97, pp. 100–109, Apr. 2014.
- [17] J. Chen, C. Richard, Y. Song, and D. Brie, "Transient performance analysis of zero-attracting LMS," *IEEE Signal Process. Lett.*, vol. 23, no. 12, pp. 1786–1790, Dec. 2016.
- [18] B. K. Das, S. Mukhopadhyay, and M. Chakraborty, "Robust adaptive filtering via convex combination of l_0 -RLS adaptive filters," in *2018 IEEE International Symposium on Circuits and Systems (ISCAS)*, 2018, pp. 1–5.
- [19] R. Tibshirani, "Regression shrinkage and selection via the LASSO," *J. Roy. Statist. Soc.*, vol. 58, no. 1, pp. 267–288, 1996.
- [20] E. Eweda, N. J. Bershad, and J. C. M. Bermudez, "Stochastic analysis of the recursive least squares algorithm for cyclostationary colored inputs," *IEEE Trans. on Signal Process.*, vol. 68, pp. 676–686, 2020.