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INFINITE BUBBLING IN NON-KÄHLERIAN GEOMETRY

GEORGES DLOUSSKY AND ANDREI TELEMAN

Abstract. In a holomorphic family $(X_b)_{b\in B}$ of non-Kählerian compact manifolds, the holomorphic curves representing a fixed 2-homology class do not form a proper family in general. The deep source of this fundamental difficulty in non-Kähler geometry is the *explosion of the area* phenomenon: the area of a curve $C_b \subset X_b$ in a fixed 2-homology class can diverge as $b \to b_0$. This phenomenon occurs frequently in the deformation theory of class VII surfaces. For instance it is well known that any minimal GSS surface X_0 is a degeneration of a 1-parameter family of simply blown up primary Hopf surfaces $(X_z)_{z\in D\setminus\{0\}}$, so one obtains non-proper families of exceptional divisors $E_z \subset X_z$ whose area diverge as $z \to 0$. Our main goal is to study in detail this non-properness phenomenon in the case of class VII surfaces. We will prove that, under certain technical assumptions, a lift $\widetilde{E}_z$ of E_z in the universal cover $\widetilde{X}_z$ does converge to an effective divisor $\widetilde{E}_0$ in $\widetilde{X}_0$, but this limit divisor is not compact. We prove that this limit divisor is always bounded towards the pseudo-convex end of $\widetilde{X}_0$ and that, when X_0 is a a minimal surface with global spherical shell, it is given by an infinite series of *compact* rational curves, whose coefficients can be computed explicitly. This phenomenon – degeneration of a family of compact curves to an infinite union of compact curves – should be called *infinite bubbling*.

We believe that such a decomposition result holds for any family of class VII surfaces whose generic fiber is a blown up primary Hopf surface. This statement would have important consequences for the classification of class VII surfaces.

MSC: 32Q65, 32Q57, 32Q55

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1. Introduction: What happens with the exceptional curves in a deformation of class VII surfaces?

1.1. **Properness properties in Kählerian and symplectic geometry.** We will use the following notation: for complex manifold X and a class $c \in H_{2d}(X,\mathbb{Z})$ we denote by $\mathcal{B}^c(X)$ the Barlet space [1] of d-cycles in X representing the class c.

Let $p : \mathcal{X} \to B$ be a family of compact n-dimensional complex manifolds parameterized by a complex manifold B. In other words, p is a proper holomorphic submersion with connected fibers. We denote by $\underline{H}_m$, $\underline{H}^m$ the locally constant sheaves (coefficient systems) on B which are associated with the presheaves $B \supset U \mapsto H_m(p^{-1}(U),\mathbb{Z})$, respectively $B \supset U \mapsto H^m(p^{-1}(U),\mathbb{Z})$. Note that the sheaf $\underline{H}^m$ coincides with the m-th

direct image $R^m p_*(\underline{\mathbb{Z}})$ of the constant sheaf $\underline{\mathbb{Z}}$ on $\mathcal{X}$. We fix $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_{2d})$, and we denote by $\mathcal{B}^e_B(\mathcal{X})$ the relative Barlet cycle space associated with e. This space is a closed subspace of the cycle space $\mathcal{B}(\mathcal{X})$ and, as a set, it coincides with the union $\cup_{b\in B}\mathcal{B}^{e_b}(X_b)$. The obvious map

$$\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$$

is holomorphic and its (reduced) fibers are precisely the spaces $\mathcal{B}^{e_b}(X_b)$. A fundamental consequence of Bishop compactness theorem states that in the Kählerian framework the projection $\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$ is also proper. More precisely:

Proposition 1.1. *Let H be a Hermitian metric on $\mathcal{X}$ such that the restrictions to the fibers of the corresponding Kähler form $\Omega_H \in A^{1,1}(\mathcal{X}, \mathbb{R})$ are all closed*[1]*. Then the map $\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$ is proper.*

Proof. The total space of the locally constant sheaf $\underline{H}^{2d} \otimes \mathbb{R} = R^{2d}p_*(\underline{\mathbb{R}})$ is a real vector bundle on B, and the assignment $b \mapsto [\Omega^d_H\big|_{X_b}]$ defines a smooth section χ of this bundle. For a cycle $C \in B^{e_b}(X_b)$ one has

$$\mathrm{vol}_H(C) = \int_C \Omega^d_H = \langle \chi(b), e_b\rangle \ .$$

Therefore the volume map $\mathrm{vol}_H : \mathcal{B}^e_B(\mathcal{X}) \to \mathbb{R}_{\geq 0}$ can be written as $\mathcal{B}^e(p)^*\langle \chi, e\rangle$, so it is bounded on $\mathcal{B}^e(p)^{-1}(K)$ for every compact set $K \subset D$. It suffices to apply the relative-compactness criterion given by Theorem 1 of [1]. □

This properness result can be intuitively interpreted as follows:

The cycles in a fixed homology class of a compact complex manifold X_b cannot disappear as X_b tends to a limit compact Kähler manifold X_{b_0}.

The properness property has a particular signification in the case $n = 2$, $d = 1$. Indeed, in this case the Barlet space $\mathcal{B}^e(X_b)$ is just a moduli space of effective divisors representing the class e, and this moduli space can be identified with the moduli space of Seiberg-Witten monopoles associated with the e-twisted canonical Spin^c-structure of X_b and a suitable perturbation of the monopole equation (see [17], [11], [16]). Therefore, in this case, the properness property above is a consequence of the general properness property – with respect to the space of parameters – for moduli spaces of Seiberg-Witten monopoles, which is crucial for proving that Seiberg-Witten invariants are well-defined.

A similar properness result – based on a version of Gromov compactness theorem – holds in symplectic geometry: let $\mathcal{X} \to B$ be a proper smooth map with connected fibers in the differentiable category, suppose that Ω is a smooth 2-form on $\mathcal{X}$ whose restrictions to the fibers $\omega_b := \Omega\big|_{X_b}$ are all symplectic forms (in particular closed), and consider a smooth family $(J_b)_{b\in B}$ of almost complex structures on the fibers, such that J_b is compatible with ω_b for any $b \in B$, and a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$. Let $(b_n)_{n\in\mathbb{N}^*}$ be a sequence in $B \setminus \{b_0\}$ converging to b_0 and consider, for every $n \in \mathbb{N}$, an almost holomorphic map

$$\varphi_n : (\Sigma, j_n) \to (X_{b_n}, J_{b_n})$$

representing $e_{b_n} \in H_2(X_{b_n}, \mathbb{Z})$, where j_n is an almost holomorphic structure on a differentiable closed surface Σ. Since the map $b \mapsto [\omega_b] \in H^2(X_b, \mathbb{R})$ defines a smooth

[1]We can always construct such a Hermitian metric in a neighborhood of a fixed fiber X_{b_0} if this fiber is Kählerian. It suffices to consider a smooth family of Kähler metrics h_b on X_b (using the openness property of the Kähler condition) for b close to b_0, and to add the pull-back of a Hermitian metric on the base.

section in the vector bundle $\underline{H}^2 \otimes \mathbb{R}$, we get a bound of the area sequence

$$\left(\int_\Sigma \varphi_n^*(\omega_{b_n}) \right)_{n\in\mathbb{N}} = \left(\langle [\omega_{b_n}], e_{b_n} \rangle \right)_{n\in\mathbb{N}} .$$

Therefore (using the terminology and the main result of [12]) there exists a subsequence (b_{n_k}) of $(b_n)_{n\in\mathbb{N}}$ such that that the sequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ converges in the $\mathcal{C}^m$-topology to a *cusp pseudo-holomorphic curve*

$$\varphi : \cup_l \Sigma_l \to X_{b_0} .$$

A similar convergence result can be proved for sequences of cusp curves. In other words, in symplectic geometry – as in Kählerian geometry – pseudo-holomorphic curves representing a fixed homology class can degenerate, but *they cannot disappear in a limit process*. An important particular case is the one when $j_n = j$ is independent of n. In this case it is known that the domain $\cup_l \Sigma_l$ of the limit cusp curve is obtained from Σ by adding a union of *bubble trees* of 2-spheres. In particular, when $(\Sigma, j) = \mathbb{P}^1$, all the irreducible components of the limit cusp curve will be 2-spheres.

These properness results and their natural generalizations for perturbed pseudo-holomorphic curves, play a crucial role in proving that (different versions of) Gromov-Witten invariants are well-defined.

For deformations of compact non-Kählerian manifolds the analogue properness results do not hold. *The properness property fails even in the cases when all cycle spaces $\mathcal{B}^e(X_b)$ are compact.* This is one of the main difficulties in non-Kählerian geometry, which has dramatic consequences, for instance: in non-Kählerian geometry one cannot use enumerative methods of curves to construct deformation invariants, and this *even if one considers only moduli spaces of curves which are finite for all manifolds in the considered deformation class!* In non-Kählerian geometry holomorphic curves representing a given homology class *can* disappear in a limit process, because the area of the curves in a given homology class is not a priori bounded, so it can explode to infinity when one approaches a fixed fiber X_{b_0}. Probably the most convincing examples which illustrates this difficulty (and its dramatic consequences) appear in the deformation theory of class VII surfaces.

1.2. **Deformations of class VII surfaces. What happens with the exceptional curves?** Let X be a class VII surface with $b_2 := b_2(X) > 0$ admitting a global spherical shell (a GSS), see section 4 for details. We will call such a surface a GSS surface. For such a surface one has $\pi_1(X, x_0) \simeq \mathbb{Z}$, and the universal cover $\widetilde{X}$ has two ends, a pseudo-convex end and a pseudo-concave end. Taking a non-separating strictly pseudo-convex embedded 3-sphere $\Sigma \subset X$, consider a lift A of $X \setminus \Sigma$ in $\widetilde{X}$ and note that $\bar{A}$ is a compact surface bounded by two lifts of Σ to $\widetilde{X}$; the pseudo-convex end of $\widetilde{X}$ corresponds to the pseudo-convex boundary component of A [2].

For such a surfaces one also has $H_2(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{b_2}$, and the intersection form on this group is standard, i.e. there exists a basis $(e_0, \ldots, e_{b_2-1})$ in $H_2(X, \mathbb{Z})$ such that $e_i \cdot e_j = -\delta_{ij}$. Changing signs if necessary one can assume that

$$-c_1(X) = c_1(\mathcal{K}_X) = \sum_{i=0}^{b_2-1} e_i ,$$

and the basis obtained in this way is unique up to order.

By results of Kato and Dloussky it is well-known that any minimal GSS surface X_0 is deformable in blown up primary Hopf surfaces. More precisely, there exists a holomorphic family $\mathcal{X} \to D$ parametrized by the unit disk $D \subset \mathbb{C}$ having X_0 as central

[2]Note that the terminology used here for the two ends of $\widetilde{X}$ is different from the one used in [4], but agrees with the terminology used in the theory of complete, non-compact complex manifolds.

fiber such that $X_z \simeq [\hat{H}_z]_{P(z)}$ is biholomorphic to primary Hopf surface $H(z)$ blown up in a finite set of simple points $P(z) = \{p_0(z), \dots, p_{b_2-1}(z)\}$. Therefore, for any $z \neq 0$, the fiber X_z has b_2 exceptional curves $E_{0,z}, \dots, E_{b_2-1,z}$ representing the classes e_i. On the other hand the central fiber X_0 is minimal hence, by a result of Nakamura (see [10], Lemma 1.1.3]), it does not admit any effective divisor at all representing a class e_i. This shows that the rational curve $E_{i,z}$ has no limit at all as $z \to 0$, even if one admits cusp curves (and bubbling trees) in the limit process.

Note that, for $z \neq 0$, the exceptional curve $E_{i,z}$ is a regular point in the moduli space of divisors representing the class e_i on X_z, so if one tries to define a Gromov-Witten type "invariant" of X_z by counting the rational curves in this homology class, the result will be 1. But the same "invariant" will be 0 for X_0.

Similar phenomena appear for deformations of minimal GSS surfaces. For instance, there exists a family $\mathcal{X} \to D$ of minimal GSS surfaces with $b_2 = 1$ whose generic fiber X_z ($z \neq 0$) has only one irreducible curve C_z, which is homologically trivial, whereas X_0 has only one irreducible curve D, which represents the class $-e_0$. In this family all curves are singular rational curves with a simple node. In both cases one can study explicitly the evolution of the area of the curves $E_{i,z}$, C_z and see that their area tends to infinity as $z \to 0$.

The conclusion is clear: one cannot hope to define deformation invariants for class VII surfaces by counting holomorphic curves in a given homology class. This is one of the major difficulties in the classification of class VII surfaces, because the structure of curves on these surfaces is related to the fundamental classification problem. Indeed, by the main result of [6] one knows that the global spherical conjecture (which, if true, would complete the classification of class VII surfaces) reduces to the conjecture:

C: *Any minimal class VII surface X with $b_2(X) > 0$ has $b_2(X)$ rational curves.*

Therefore, proving existence of curves is a fundamental problem in the theory of class VII surfaces. Unfortunately the examples above show that one cannot hope to prove such existence results using invariants of Gromov-Witten type. Note however that, taking into account this difficulty (the lack of Gromov-Witten type invariants), the conjecture above becomes quite intriguing. Indeed, this conjecture (which is true for all GSS surfaces) implies that the total number of rational curves *is* an invariant for class VII surfaces, although this invariant cannot be obtained as the sum of Gromov-Witten type invariants defined for individual homology classes (because, as we have just seen, the number of rational curves in an individual class is *not* an invariant). In other words, although the homology classes represented by rational curves change in a holomorphic family, the total number of rational curves remains always constant in deformations of known class VII surfaces. In particular, when (in a limit process) a homology class loses a rational curve, there should be always another class which gets one. This "global compensation phenomenon" between different moduli spaces of rational curves is not understood yet, so conjecture **C** is not known yet even for surfaces which are degenerations of the known GSS surfaces.

Although the cause of the "curve vanishing " (non-properness) phenomenon is clear – explosion of the area in the limit process – one can still wonder when contemplating the first family described above:

"What happens with the exceptional curves $E_{i,z}$ when $z \to 0$?"

The authors have many times been asked this question when they gave talks on class VII surfaces in the past. The easy answer we have always given

"They just disappear, because their area explodes",

although perfectly correct, has never seemed to fully satisfy the audience. Therefore we decided to investigate in detail this phenomenon trying to understand what happens geometrically with the exceptional curve $E_{i,z}$ as it gets bigger and bigger.

We came to the idea of an "infinite bubbling tree" empirically, when the second author noticed that recent results of the first author [5] have an intriguing consequence: Although for a minimal GSS surface X_0 the homology class e_i is not represented by an effective divisor (as we saw above), any of its lifts $\widetilde{e}_i$ in the universal cover $\widetilde{X}_0$ is represented – in the homology with closed supports – by an infinite series of compact curves. Moreover, in every case we obtained a well defined series representing the given lift, and we saw that always this series escapes to infinity towards the pseudo-concave end of $\widetilde{X}_0$. Therefore we came to idea to prove that (although $E_{i,z}$ accumulates to the whole X_0 as $z \to 0$ by the Remmert-Stein theorem) a fixed lift $\widetilde{E}_{i,z}$ of $E_{i,z}$ in the universal covering $\widetilde{X}_z$ of X_z behaves in a controllable way as $z \to 0$, namely that

Theorem 1.2. *The family of rational curves $(\widetilde{E}_{i,z})_{z\in D\setminus\{0\}}$ fits in a flat family $\widetilde{\mathcal{E}}_i$ of effective divisors over the whole base D. In particular this family has a limit as $z \to 0$, which is given by an infinite series of compact curves which escapes to infinity towards the pseudo-concave end of $\widetilde{X}_0$.*

The main purpose of this article is to prove this result. More precisely we will prove that the limit given by the theorem is precisely the series representing $\widetilde{e}_i$ we discovered empirically. We called this phenomenon "infinite bubbling", and we believe that this happens in all deformations of class VII surfaces when area explosion occurs. In fact we will prove a more general result:

Theorem 1.3. *Let $\mathcal{X} \to \Delta$ be any holomorphic family of GSS surfaces with $b_2 > 0$, parameterized by the standard ball $\Delta \subset \mathbb{C}^r$. Then for any lift $\widetilde{e}_i$ of a homology class e_i to the universal cover $\widetilde{\mathcal{X}}$, there exists an effective divisor $\widetilde{\mathcal{E}}_i \subset \widetilde{\mathcal{X}}$ flat over Δ such that*

1. *The fiber $\widetilde{E}_{i,z} \subset \widetilde{X}_z$ of $\widetilde{\mathcal{E}}_i$ over a point $z \in \Delta$ is a lift of the exceptional curve E_i representing e_i, for every $z \in \Delta$ for which X_z contains such an exceptional curve.*
2. *If X_z does not admit any exceptional effective divisor in the class e_i then the corresponding fiber $\widetilde{E}_{i,z}$ of $\widetilde{\mathcal{E}}_i$ is a series of compact curves which escapes to infinity towards the pseudo-concave end of $\widetilde{X}_z$. This divisor represents the image of the class $\widetilde{e}_i$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(X, \mathbb{Z})$.*

Finally, we can generalize this result for deformations of an *unknown* minimal class VII surface which can be deformed in GSS surfaces:

Theorem 1.4. *Let $\mathcal{X} \to \Delta$ be a holomorphic family such that X_z is a GSS surface with $b_2 > 0$ for any $z \in \Delta \setminus A$, where $A \subset \Delta$ is an analytic subset of codimension ≥ 2, and X_a is minimal for every $a \in A$. Then for any lift $\widetilde{e}_i$ of a homology class e_i to the universal cover $\widetilde{\mathcal{X}}$, there exists an effective divisor $\widetilde{\mathcal{E}}_i \subset \widetilde{\mathcal{X}}$ flat over Δ such that the fiber $\widetilde{E}_{i,z} \subset \widetilde{X}_z$ of $\widetilde{\mathcal{E}}_i$ over a point $z \in \Delta$ has the properties:*

1. *it represents the image of the class $\widetilde{e}_i$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(X, \mathbb{Z})$ for every $z \in \Delta$,*
2. *is a lift of the exceptional curve E_i representing e_i, for every $z \in \Delta$ for which X_z contains such an exceptional curve.*

Unfortunately we did not succeed to prove that the second statement of Theorem 1.3 also holds in this case, so we cannot prove that, for $a \in A$, the limit divisor $\widetilde{E}_{i,a}$ is given by a series of *compact* curves. The difficulty is to rule out the appearance of an open Riemann surface in the limit. Such a result would be of crucial importance for the classification of class VII surfaces, because it will prove the main conjecture **C** for any class VII surface which fits as the central fiber of a bidimensional family $\mathcal{X} \to \Delta \subset \mathbb{C}^2$ whose fibers X_z, $z \neq 0$ are GSS surfaces.

2. The non-minimality divisors

A compact effective divisor D on a (non-necessary compact) complex surface X will be called *simply exceptional* if there exists a *smooth* surface Y, a point $y \in Y$, and a surjective holomorphic map $c : X \to Y$ such that $c_{|X\setminus D} : X \setminus D \to Y \setminus \{y\}$ is biholomorphic, and the fiber $c^{-1}(y)$ over y is D (regarded as a complex subspace of X). If this is the case, the pair (X, D) can be obtained from Y by iterated blowing up at y. More precisely (X, D) is the last term (X_n, D_n) in a sequence $(X_1, D_1), \dots, (X_n, D_n)$ $(n \geq 1)$, where

- X_1 is the blow up of Y at y and D_1 is the corresponding exceptional divisor,
- X_k is obtained from X_{k-1} by blowing up at a point $y_{k-1} \in D_{k-1}$ for $2 \leq k \leq n$, and D_k is the pre-image of D_{k-1} in X_k.

Therefore D is the pre-image in X_n of the exceptional curve of first kind $D_1 \subset X_1$, and it is a tree of smooth rational curves containing at least one exceptional curve of the first kind.

Remark 2.1. *For simply exceptional divisor $D \subset X$ one has*

$$(D, \mathcal{K}_X) = (D, \mathcal{N}_{D/X}) = -1 \ .$$

Using the duality isomorphism $PD : H_2(X, \mathbb{Z}) \simeq H^2_c(X, \mathbb{Z})$ we have

$$D^2 := \langle PD([D]), [D] \rangle = -1 \ .$$

Since the arithmetic genus $g_a(D)$ vanishes, one obtains $h^0(\mathcal{O}_D(D)) - h^1(\mathcal{O}_D(D)) = 0$ by Riemann-Roch theorem for embedded curves ([2], p. 65). Therefore, since $\deg_D(\mathcal{O}_D(D)) = -1 < 0$, we get

$$h^0(\mathcal{O}_D(D)) = h^1(\mathcal{O}_D(D)) = 0 \ ,$$

so, denoting by $e \in H_2(X, \mathbb{Z})$ the homology class represented by D, it follows that the Douady space $\mathcal{D}ou^e(X)$ of compact effective divisors representing this class is 0-dimensional at D, and D is a smooth point in this space (see [3] p. 135).

Remark 2.2. *Let X be a compact surface and D, E simple exceptional divisors representing the same rational homology class $e \in H_2(X, \mathbb{Q})$. Then $D = E$.*

Proof. (X, D) is the last term (X_n, D_n) of a sequence of pairs $(X_1, D_1), \dots, (X_n, D_n)$ obtained from (Y, y) as above. We prove the statement by induction on n. If $n = 1$, D is an exceptional curve of the first kind. Since $DE = -1$, it follows that D is one of the irreducible components of E. But the $\mathbb{Q}$-homology classes of the irreducible components of a simply exceptional divisor are linearly independent, so one can have $[D]_{\mathbb{Q}} = [E]_{\mathbb{Q}}$ if and only if $D = E$.

If $n > 1$, let D' be the exceptional curve of the first kind of the blowing up $X_n \to X_{n-1}$. One has obviously $D'D = 0$, so $D'E = 0$. Therefore either D' is an irreducible component of E, or $D' \cap E = \emptyset$. In both cases the direct images $(p_n)_*(D)$, $(p_n)_*(E)$ via the contraction map $p_n : X_n \to X_{n-1}$ are simply exceptional divisor in the rational homology class $(p_n)_*(e)$, so they coincide by induction assumption. On the other hand one can write $D = p_n^*((p_n)_*(D))$, $E = p_n^*((p_n)_*(E))$, so $D = E$. □

The condition "X admits a simply exceptional effective divisor D" is open with respect to deformations of X. More precisely:

Proposition 2.3. *let $p : \mathcal{X} \to B$ be a holomorphic submersion with $\dim(\mathcal{X}) = \dim(B) + 2$. Let $D_0 \subset X_{b_0}$ be a simply exceptional divisor, and $e \in H_2(\mathcal{X}, \mathbb{Z})$ its homology class in $\mathcal{X}$. There exists a neighborhood $\mathcal{U}$ of D_0 in $\mathcal{D}ou^e(\mathcal{X})$ such that*

1. *for every $D \in \mathcal{U}$, $p(D)$ is a singleton denoted $\{\mathfrak{p}(D)\}$ in B and D is a tree of rational curves in $X_{\mathfrak{p}(D)}$,*
2. *the induced map $\mathfrak{p} : \mathcal{U} \to B$ is a local biholomorphism at D_0,*

(3) *D is simply exceptional in $X_{\mathfrak{p}(D)}$ for every $D \in \mathcal{U}$*

Proof. Let $V \subset B$ an open neighborhood of b_0 which is the domain of a chart, and $\mathcal{V} \subset \mathcal{D}ou^e(\mathcal{X})$ the neighborhood of D_0 in $\mathcal{D}ou^e(\mathcal{X})$ consisting of 1-dimensional complex subspaces of $\mathcal{X}$ which represent the class e and are contained in $p^{-1}(V)$.

For every $D \in \mathcal{V}$ the projection $p(D) \subset V$ is a finite union of points. On the other hand it's easy to see that any small flat deformation of a tree of smooth rational curves is again a tree of smooth rational curves, in particular it is connected. Therefore, replacing $\mathcal{V}$ by a smaller neighborhood if necessary, the conclusion (1) will be fulfilled.

The normal bundle $\mathcal{N}_{\mathcal{X}}(D_0)$ of D_0 in $\mathcal{X}$ fits in a short exact sequence on D_0:

$$0 \to \mathcal{N}_{X_{b_0}}(D_0) = \mathcal{O}_{D_0}(D_0) \to \mathcal{N}_{\mathcal{X}}(D_0) \to D_0 \times T_{b_0}(B) \to 0 \ .$$

Since $h^1(\mathcal{O}_{D_0}(D_0)) = h^1(\mathcal{O}_{D_0}) = 0$ we obtain

$$H^0(\mathcal{N}_{\mathcal{X}}(D_0)) = T_{b_0}(B) \ , \ H^1(\mathcal{N}_{\mathcal{X}}(D_0)) = \{0\} \ .$$

This shows that $\mathcal{D}ou^e(\mathcal{X})$ is smooth at D_0 and the projection on B defined by $\mathfrak{p}$ is a local biholomorphism at D_0. This proves (2). Let $\mathcal{U} \subset \mathcal{V}$ be an open neighborhood of D_0 on which $\mathfrak{p}$ is biholomorphic, and put $U := \mathfrak{p}(\mathcal{U}) \subset B$. We obtain a holomorphic family of trees of rational curves $(D_b)_{b\in U}$ with $D_{b_0} = D_0$.

The third statement can be proved by induction with respect to the number m of irreducible components of D_0. Consider an irreducible component D_m of D_0 which is an exceptional curve of the first kind. D_m is itself a simply exceptional divisor, so the statements (1), (2) apply to D_m, and, taking $\mathcal{U}$ and U sufficiently small, we obtain a holomorphic family

$$(D_{m,b})_{b\in U} \ , \ D_{m,b} \subset X_b$$

of simply exceptional divisors. But, by the stability theorem of Kodaira, any small deformation of an exceptional curve of the first kind is again an exceptional curve of the first kind, so we may assume that all $D_{m,b}$ are all exceptional curves of the first kind. This proves the statement when $m = 1$. Suppose $m > 1$. Contracting in X_b the curve $D_{m,b}$ to a point y_b for every $b \in B$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_U & \stackrel{\pi}{\longrightarrow} & \mathcal{Y} \\ & {\scriptstyle p_U}\searrow \quad \swarrow{\scriptstyle s} & \\ & U & \end{array}$$

where π is proper and bimeromorphic whose restriction to any fiber X_b is just the blowing down map $\pi_b : X_b \to Y_b$ which contracts $D_{m,b} \subset X_b$ to y_b. Let $\mathcal{D}$, $\mathcal{D}_m \subset \mathcal{X}_U$ be the classifying divisors of the families $(D_b)_{b\in U}$, $(D_{m,b})_{b\in U}$. Supposing U connected, we see that $\mathcal{D}_m$ is irreducible. One has $(\mathcal{D} \cap \mathcal{D}_m) \cap X_{b_0} = D_{m,b_0}$, so (since we supposed $m > 1$) one has $\mathcal{D} \neq \mathcal{D}_m$. Let $\mathcal{D}'$ be the divisor which coincides with $\mathcal{D}$ if $\mathcal{D}_m \not\subset \mathcal{D}$ and is obtained from $\mathcal{D}$ by removing the $\mathcal{D}_m$ if $\mathcal{D}_m \subset \mathcal{D}$. The projection on $\mathcal{Y}$ of every irreducible component of $\mathcal{D}'$ is reduced and irreducible of codimension 1, so $\mathcal{D}'' = \pi(\mathcal{D}')$ is an effective divisor of $\mathcal{Y}$. Any fiber D''_b of this divisor over a point $b \in U$ is an effective divisor of Y_b (more precisely a tree of rational curves in Y_b) so $\mathcal{D}''$ is flat over U by Lemma 2.4 below. By induction assumption all fibers D''_b are simply exceptional divisors. Note now that D_b is the pre-image $\pi_b^*(D''_b)$ of D''_b in X_b. Therefore D_b will be simply exceptional, too. □

Lemma 2.4. *Let $p : \mathcal{X} \to B$ be a submersion of complex manifolds with* $\dim(\mathcal{X}) = n$, $\dim(B) = m$. *Let $\mathcal{D} \subset \mathcal{X}$ be an effective divisor such that $\mathcal{D} \cap X_b$ is a (possibly empty) effective divisor of the fiber X_b for every $b \in B$. Then $\mathcal{D}$ is flat over B.*

Proof. Let $x \in \mathcal{D}$ and let $\phi : U \to \mathbb{C}$ be a local equation for $\mathcal{D}$ around x, i.e. U is an open neighborhood of x in $\mathcal{X}$ and $\mathcal{D} \cap U = Z(\phi)$. Consider the map $\psi := (p, \phi) : U \to B \times \mathbb{C}$,

and note that the fiber $\psi^{-1}(\psi(x))$ of this map can be identified with $\mathcal{D}\cap U\cap X_{p(x)}$ so the dimension at x of this fiber has minimal value $n-m-1$. By the fiber dimension semicontinuity theorem we see that, replacing U with a smaller open neighborhood if necessary, we may assume that $\dim_\xi \psi^{-1}(\psi(\xi)) = n-m-1$ for every $\xi\in U$. Therefore ψ will be flat by the flatness criterion given by the Corollary on p. 158 in [8]. Now we apply the base change property of the flat morphisms to the base change $B\ni b\mapsto (b,0)\in B\times\mathbb{C}$ and we get that $\mathcal{D}\cap U$ is flat over B. □

For a holomorphic family $\mathcal{X}\to B$ of compact complex surfaces we will denote by $k=(k_b)_{b\in B}\in H^0(B,\underline{H}^2)$ the section defined by the classes $k_b:=c_1(\mathcal{K}_{X_b})$, $b\in B$.

Corollary 2.5. *Let $\mathcal{X}\to B$ be a holomorphic family of compact complex surfaces, and let $e=(e_b)_{b\in B}\in H^0(B,\underline{H}_2)$ such that $\langle e,k\rangle=-1$ and $e\cdot e=-1$ with respect to the intersection form induced from any $H_2(X_b,\mathbb{Z})$. Then the subset*

$$B_e:=\{b\in B|\ X_b \text{ admits a simply exceptional divisor representing the class } e_b\}\subset B$$

is open with respect to the classical topology.

Lemma 2.6. *Let S be a class VII surface and let C be an effective divisor in X with*

$$C^2=(C,\mathcal{K}_S)=-1\ .$$

Then $C=E+\Gamma$, where E is a simply exceptional divisor, $p\geq 0$, and $\Gamma=f^\star\Gamma'$ is the pre-image of a homologically trivial effective divisor $\Gamma'\subset S'$ via the contraction $f:S\to S'$ to the minimal model.

Proof. If the minimal model of S is an Inoue surface S' with $b_2(S')=0$, the result is clear, because an Inoue surface with $b_2=0$ does not contain any curve. Therefore we shall suppose that the minimal model is a Hopf surface or a minimal class VII surface with $b_2>0$. Let $C=\sum_{i=1}^m n_iC_i$ $(n_i>0)$ be the decomposition of C as linear combination of irreducible components. We prove the result by induction on the sum $\sigma_C:=\sum_{i=1}^m n_i\geq 1$. If $\sigma_C=1$, then $C=C_1$ is irreducible with $C_1^2=(C,\mathcal{K}_S)=-1$, so it is an exceptional curve of the first kind by Proposition 2.2 in [2].

Suppose now $\sigma_C\geq 2$. We have

$$-1=(C,\mathcal{K}_S)=\sum_i n_i(C_i,\mathcal{K}_S)$$

hence there exists an index j such that $(C_j,\mathcal{K}_S)<0$. Since the intersection form of a class VII surface is negative definite, one has $C_j^2\leq 0$, with equality if and only if C_j is homologically trivial. This would contradict $(C_j,\mathcal{K}_S)<0$, so necessary $C_j^2<0$. By Proposition 2.2 in [2] again it follows that C_j is an exceptional curve of the first kind.

Case 1. The homology classes $[C]_{\mathbb{Q}}$, $[C_j]_{\mathbb{Q}}$ coincide in $H_2(S,\mathbb{Q})$. In this case the effective divisor $\Gamma:=C-C_j$ is $\mathbb{Q}$-homologically trivial. Therefore the effective divisor $\Gamma':=f_*(\Gamma)\subset S'$ is $\mathbb{Q}$-homologically trivial, too. We can write $\Gamma=f^*(\Gamma')+F$, where F is an effective divisor contained in the exceptional divisor of the contraction map f. But F must be also $\mathbb{Q}$-homologically trivial, so it is empty. This shows that $\Gamma=f^*(\Gamma')$, where Γ' is a $\mathbb{Q}$-homologically trivial effective divisor in the minimal model S'. But, using Enoki's theorem concerning the classification of class VII surfaces admitting a numerically trivial divisor, it follows that on such a surface any $\mathbb{Q}$-homologically trivial effective divisor is homologically trivial.

Case 2. The homology classes $[C]_{\mathbb{Q}}$, $[C_j]_{\mathbb{Q}}$ are different. We contract the exceptional curve C_j obtaining a blowing down map $g:S\to S_0$. The homology group $H_2(S,\mathbb{Z})$ has a decomposition

$$H_2(S,\mathbb{Z})=\mathbb{Z}[C_j]\oplus H^\perp\ ,$$

which is orthogonal with respect to the intersection form q_S of S, and g_* induces an isomorphism

$$(H^\perp, q_S|_{H^\perp\times H^\perp}) \xrightarrow{\simeq g_*|_{H^\perp}} (H_2(S_0,\mathbb{Z}), q_{S_0}) \tag{1}$$

Since the rational intersection form $q_S^{\mathbb{R}}$ is negative definite, we have

$$0 \leq |CC_j| = \left|q_S^{\mathbb{R}}([C]_{\mathbb{R}}, [C_j]_{\mathbb{R}})\right| \leq \frac{1}{2}(-C^2 - C_j^2) = 1\ , \tag{2}$$

with equality if and only if $[C]_{\mathbb{R}}, [C_j]_{\mathbb{R}}$ are colinear. But since these classes belong to the lattice $\mathrm{im}(H_2(S,\mathbb{Z}) \to H_2(S,\mathbb{R}))$ and $(C_j, \mathcal{K}_S) = (C, \mathcal{K}_S) = -1$, the colinearity condition implies $[C]_{\mathbb{R}} = [C_j]_{\mathbb{R}}$, which cannot hold in our case. Therefore the inequality (2) implies $CC_j = 0$, i.e. $[C] \in H^\perp$. Using the isomorphism (1) we see that $g_*(C)^2 = C^2 = -1$. On the other hand $\mathcal{K}_S = g^*(\mathcal{K}_{S_0})(C_j)$, so

$$-1 = (C, \mathcal{K}_S) = (C, g^*(\mathcal{K}_{S_0})(C_j)) = CC_j + (C, g^*(\mathcal{K}_{S_0})) = (g_*(C), \mathcal{K}_{S_0})\ .$$

This shows that the effective divisor $C_0 := g_*(C)$ of S_0 satisfies the conditions

$$C_0^2 = (C_0, \mathcal{K}_{S_0}) = -1\ .$$

Since $\sigma_{C_0} < \sigma_C$ we can apply the induction assumption to C_0. Note that $C = g^*(C_0)$. To see this it suffices to see that

- they differ by a multiple of C_j,
- the corresponding homology classes coincide, because they belong to $H^\perp$ and have the same image via g_*.

By induction assumption we know that C_0 decomposes as $C_0 = E_0 + f_0^*(\Gamma_0)$, where $f_0 : S_0 \to S'$ is the contraction to the minimal model of S_0 (which is also the minimal model of S), and Γ_0 is a homologically trivial divisor in S_0. Using the equality $C = g^*(C_0)$, we get $C = g^*(E_0) + g^*(f_0^*(\Gamma_0))$. It suffices to note that $g^*(E_0)$ is a simply exceptional divisor. □

The minimal class VII surfaces admitting a (non-empty) homologically trivial effective divisor Γ are classified [7], we can describe explicitly the second term $f^*(\Gamma)$ in the decomposition given by Lemma 2.6.

Remark 2.7. *If Γ is a (non-empty) homologically trivial effective divisor on a minimal class VII surface X, then the pair (X, Γ) is one of the following:*

1. *X is an elliptic Hopf surface and $\Gamma = \sum_{i=1}^k n_i C_i$, where $n_i \geq 0$, $\sum_{i=1}^k n_i > 0$ and C_i are elliptic curves,*
2. *X is a Hopf surface with two elliptic curves C_1, C_2, and $\Gamma = n_1C_1 + n_2C_2$ with $n_i \geq 0$, $n_1 + n_2 > 0$,*
3. *X is a Hopf surface with an elliptic curve C and $\Gamma = nC$ with $n > 0$,*
4. *X is an Enoki surface and $\Gamma = nC$, where $n > 0$ and $C \subset X$ is a cycle of rational curves.*

Lemma 2.8. *Let $\mathcal{X} \to B$ be a holomorphic family of class VII surfaces, let $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$, $b_0 \in B$ and $(b_n)_{n\in\mathbb{N}^*}$ a sequence in B converging to b_0. Consider, for every $n \in \mathbb{N}^*$, a simply exceptional divisor $D_n \subset X_{b_n}$ in the class $e_{b_n} \in H_2(X_{b_n}, \mathbb{Z})$ such that the sequence $(D_n)_{n\in\mathbb{N}^*}$ converges in $\mathcal{B}_B^e(\mathcal{X})$ to a 1-cycle $D_0 \subset X_{b_0}$. Then*

1. *The limit cycle D_0 is a simply exceptional divisor in X_{b_0}.*
2. *Regarding D_n, D_0 as complex subspaces of $\mathcal{X}$, one has $\lim_{n\to\infty} D_n = D_0$ in the Douady space $\mathcal{D}ou(\mathcal{X})$.*

Proof. By Lemma 2.6 we see that D_0 decomposes as $D_0 = E_0 + f_0^*(\Gamma_0)$, where E_0 is a simply exceptional divisor in the class e, $f_0 : X_{b_0} \to X'_{b_0}$ is the contraction to the minimal model, and Γ_0 is a homologically trivial effective divisor in X'_{b_0}. By Proposition 2.3 there exists an open neighborhood $\mathcal{U}$ of E_0 in $\mathcal{D}ou(\mathcal{X})$ such that every

element $E \in \mathcal{E}$ is a simply exceptional divisor in the fiber $X_{\mathfrak{p}(D)}$ where $\mathfrak{p} : \mathcal{U} \to U$ is a biholomorphism on an open neighborhood U of b_0. For sufficiently large n one has $b_n \in U$ we obtain a simply exceptional divisor $E_n := \mathfrak{p}^{-1}(b_n)$ in X_{b_n}, and obviously

$$\lim_{n\to\infty} E_n = E_0$$

in the Douady space $\mathcal{D}ou(\mathcal{X})$, so also in the Barlet space $\mathcal{B}^e_B(\mathcal{X})$. But, for every $n \in \mathbb{N}^*$, D_n and E_n are simply exceptional divisors representing the same homology class, so by Lemma 2.2 we have $D_n = E_n$ for any $n \in \mathbb{N}^*$. But the Barlet cycle-space is Hausdorff so, since the limit of a convergent sequence in a Hausdorff space is unique, $D_0 = E_0$. This shows that D_0 is a simply exceptional divisor, and the that $\lim_{n\to\infty} D_n = D_0$ also holds in the Douady space. □

Let $p : \mathcal{X} \to B$ be a deformation of class VII surfaces parameterized by a complex manifold B. Let again $k \in H^0(B, \underline{H}^2)$ be the element defined by the family of Chern classes $k_b := c_1(\mathcal{K}_{X_b})$, $b \in B$. Using the exponential short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on $\mathcal{X}$, the vanishing of the cohomology groups $H^2(X_b, \mathcal{O}_{X_b})$ we obtain the exact sequence of sheaves on B

$$0 \to {R^1p_*(\mathcal{O}_\mathcal{X})}/{\underline{H}^1} \to R^1p_*(\mathcal{O}^*_\mathcal{X}) \to \underline{H}^2 \to 0 \ . \tag{3}$$

The quotient sheaf on the left (respectively the sheaf in the middle) is the sheaf of holomorphic sections in a locally trivial complex Lie group fiber bundle Pic^0 (respectively Pic) over B whose fiber over $b \in B$ is $\mathrm{Pic}^0(X_b)$ (respectively $\mathrm{Pic}(X_b)$).

On a class VII surface the canonical morphism $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$ is an isomorphism, so $R^1p_*(\mathcal{O}_\mathcal{X})$ can be identified with $\underline{H}^1 \otimes \mathcal{O}_B$. It is easy to see that the sheaf $\underline{H}^1$ (which is obviously locally constant with fiber isomorphic to $\mathbb{Z}$) is in fact constant; it suffices to see that the degree maps associated with a smooth family of Gauduchon metrics on the fibers define canonical orientations of the real lines $H^1(X_b, \mathbb{R}_{>0}) \subset H^1(X_b, \mathbb{C}^*)$, so canonical orientations of the lines $H^1(X_b, \mathbb{R})$, so canonical generators of the infinite cyclic groups $H^1(X_b, \mathbb{Z})$. Therefore the complex Lie group bundle Pic^0 is in fact trivial with fiber $\mathbb{C}^*$. It is convenient to choose a trivialization $\phi = (\phi_b : \mathbb{C}^* \to \mathrm{Pic}^0(X_b))_{b\in B}$ of this fiber bundle such that for every $b \in B$ one has

$$\lim_{z\to 0} \deg_g(\phi_b(z)) = \infty$$

with respect to a Gauduchon metric on a fiber X_b.

For every section $c \in H^0(B, \underline{H}^2)$ we obtain a subbundle Pic^c of Pic, which becomes a principal $\mathbb{C}^*$-bundle via the chosen trivialization $\phi : B \times \mathbb{C}^* \to \mathrm{Pic}^0$. The associated line bundle $\mathcal{P}^c$ on B can be obtained by adding formally to Pic^c a zero section $\{0_b |\ b \in B\}$, such that for every $b \in B$ one has

$$\forall \lambda \in \mathrm{Pic}^{c_b}(X_b) \quad \lim_{z\to 0} \phi_b(z) \otimes \lambda = 0_b \ .$$

Theorem 2.9. *Let $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ such that $e^2 = \langle e, k\rangle = -1$, and let B_e be the open subset of B consisting of points b for which X_b admits a simply exceptional divisor E_b in the class e_b. If $B_e \neq \emptyset$ then*

1. *There exists an effective divisor $\mathcal{E}_0 \subset p^{-1}(B_e)$ flat over B_e whose fiber over a point $b \in B_e$ is E_b.*
2. *The map $\sigma_e : B_e \to \mathcal{P}^e$ defined by $b \mapsto [\mathcal{O}_{X_b}(E_b)]$ extends to a holomorphic section s_e of the line bundle $\mathcal{P}^e$ over B.*
3. *One has $B_e = B \setminus Z(s_e)$, where $Z(\cdot)$ stands for the vanishing locus of a section. In particular, the complement of B_e is a divisor $H_e := Z(s_e)$, so B_e is Zariski open.*

Proof. Using Remark 2.2 and Proposition 2.3 we see that the map

$$B_e \ni b \mapsto E_b \in \mathcal{D}ou(\mathcal{X})$$

is well-defined and holomorphic. The first statement follows now from the universal property of the Douady space.

Let $(g_b)_{b\in B}$ be a smooth family of Gauduchon metrics on the fibers X_b normed such that for every $b \in B$ it holds $\deg_{g_b}(\phi_b(z)) = -\log|z|$. The map $\mathcal{P}^e \to [0,\infty)$ given by

$$\mathrm{Pic}^e(X_b) \ni [\mathcal{L}] \mapsto e^{-\deg_{g_b}(\mathcal{L})}\ ,\ 0_b \mapsto 0$$

defines a norm $\|\cdot\|$ on the holomorphic line bundle $\mathcal{P}^e$ on B. We claim

Claim: For every $b_0 \in \bar{B}_e \setminus B_e$ one has $\lim_{b\to b_0}\|\sigma_e(b)\| = 0$.

Indeed, if not, there exists a sequence $(b_n)_{n\in\mathbb{N}^*}$ in B_e converging to b_0 and $\varepsilon > 0$ such that $\|\sigma_e(b_n)\| \geq \varepsilon$ for every $n \in \mathbb{N}^*$. This implies that the sequence

$$(\deg_{g_{b_n}}(\sigma_e(b_n)))_{n\in\mathbb{N}^*} = (\mathrm{vol}_{g_{b_n}}(E_{b_n}))_{n\in\mathbb{N}^*}$$

is bounded, so there exists a subsequence of $(E_{b_n})_{n\in\mathbb{N}^*}$ converging in the Barlet space $\mathcal{B}^e_B(\mathcal{X})$ [1]. By Lemma 2.8 the limit divisor E_0 in X_{b_0} will be simply exceptional, so $b_0 \in B_e$, which contradicts the choice of b_0. This proves the claim.

We define now a section $s_e : B \to \mathcal{P}^e$ by

$$s_e(b) := \begin{cases} \sigma_e(b) & \text{if} \quad b \in B_e \\ 0_b & \text{if} \quad b \notin B_e \end{cases}\ .$$

Using the claim proved above we see that s_e is continuous. The first statement of the theorem follows now from Rado's theorem, and the second follows from the first and the explicit construction of the section s_e. □

Remark 2.10. *The functor which associates to a pair $(p : \mathcal{X} \to B, e)$ consisting of a holomorphic family of class VII surfaces and a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ with*

$$e^2 = \langle e, k\rangle = -1$$

the (possibly empty) effective divisor $H_e \subset B$, commutes with base change. Therefore this functor should be interpreted as an effective Cartier divisor in the moduli stack classifying pairs (X,η) consisting of a class VII surface X and a class $\eta \in H_2(X,\mathbb{Z})$ with $\eta^2 = \langle \eta, \mathcal{K}_X\rangle = -1$.

Remark 2.11. *The underlying codimension 1 analytic set $\mathfrak{H}_e \subset B$ associated with H_e coincides with the hypersurface defined by Dloussky in* [5].

3. Extension theorems

Let $p : \mathcal{X} \to B$ be a holomorphic family of class VII surfaces. Using the notations and the definitions introduced in the previous section, fix a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ such that $e^2 = (e,k) = -1$, and let $B_e := B \setminus H_e \subset B$ the associated Zariski open subset (see Theorem 2.9). We know that H_e is the vanishing divisor of a holomorphic section $s_e \in H^0(B, \mathcal{P}^e)$, which on B_e is given by $b \mapsto [\mathcal{O}_{X_b}(E_b)]$, E_b being the unique simply exceptional divisor on X_b representing the class e_b.

As we explained in the introduction the fundamental problem studied in this article is the evolution of the divisor E_b as b tends to a point $b_0 \in H_e$. Since the volume of E_b with respect to a smooth family of Gauduchon metrics on the fibers tends to infinity as $b \to b_0$, we cannot expect to understand this evolution using standard complex geometric tools applied to the given family.

In this section we will prove that, under certain assumptions, a lift $\widetilde{E}_b$ of E_b in the universal cover $\widetilde{X}_b$ of X_b does have a limit as $b \to b_0$, and this limit is a non-compact divisor of $\widetilde{X}_{b_0}$. Moreover, the family $(\widetilde{E}_b)_{b\in B}$ extends to a flat family of divisors over

the whole base B. We can prove this type of extensions results under two types of assumptions. First we will assume that all the fibers X_b are class VII surfaces with the topological type of class VII surfaces having a non-separating strictly pseudo-convex embedded 3-sphere; second we will suppose that the subset $B_0 \subset B$ consisting of points b for which X_b is an unknown class VII surface is contained in an analytic subset of codimension ≥ 2.

The diffeomorphism type of a GSS surface with $b_2 = n$ is $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$, so if the GSS conjecture was true this would be the diffeomorphism type of *any* minimal class VII surface with $b_2 = n > 0$. Moreover, by a result of Nakamura [9], any minimal class VII surface with $b_2 = n > 0$ containing a cycle of curves is a degeneration ("big deformation") of a family of blown up primary Hopf surfaces, so it also has the diffeomorphism type of a GSS surface. By the results of [14], [15] every minimal class VII surface with $b_2 \in \{1, 2\}$ does contain a cycle, so by Nakamura's result, it has the same diffeomorphism type. Therefore, studying class VII surfaces with the diffeomorphism (or, more generally, homeomorphism) type $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$, and studying deformations of such surfaces, is an important problem.

In general, if M is a topological 4-manifold M with this topological type, then every 2-homology class $e \in H_2(M, \mathbb{Z})$ is represented by an embedded surface with simply connected components, in particular it lifts to $H_2(\widetilde{M}, \mathbb{Z})$. More precisely, $H_2(\widetilde{M}, \mathbb{Z})$ is naturally an n-dimensional free $\mathbb{Z}[F]$-module, where F stands for the multiplicative group $\mathrm{Aut}_M(\widetilde{M}) = \{f^k |\ k \in \mathbb{Z}\} \simeq \mathbb{Z}$, and the quotient

$$H_2(\widetilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}[F]} \mathbb{Z} = {}^{H_2(\widetilde{M}, \mathbb{Z})}\!/_{(\mathrm{id} - f)H_2(\widetilde{M}, \mathbb{Z})}$$

is naturally isomorphic to $H_2(M, \mathbb{Z})$. Here $\mathbb{Z}$ is regarded as a $\mathbb{Z}[F]$-algebra via the trace morphism

$$\sum_{n \in \mathbb{Z}} a_n f^n \mapsto \sum_n a_n \ ,$$

and $(\mathrm{id} - f)$ stands for the ideal $(\mathrm{id} - f) = (\mathrm{id} - f)\mathbb{Z}[F] \subset \mathbb{Z}[F]$ generated by $\mathrm{id} - f$.

Let $p : \mathcal{M} \to B$ be a locally trivial topological fiber bundle with $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$ as fiber and 2-connected basis B, $\alpha : \widetilde{\mathcal{M}} \to \mathcal{M}$ the universal cover of the total space, and $\widetilde{p} := p \circ \alpha : \widetilde{\mathcal{M}} \to B$ the induced fiber bundle. Since B is 2-connected, the homotopy long exact sequence associated with the locally trivial fiber bundle p shows that the fiber embedding $\iota_b : M_b \hookrightarrow \mathcal{M}$ induces an isomorphism $\pi_1(M_b, m) \xrightarrow{\simeq} \pi_1(\mathcal{M}, m)$, so the induced projection $\alpha_b : \widetilde{M}_b \to M_b$ is a universal cover of the fiber M_b for every $m \in M_b$. Note also that the assumption "B is 2-connected" implies that the locally constant sheaves $\underline{H}_i$, $\underline{H}^i$ on B are constant, for every $i \in \mathbb{N}$. Denoting by $\underline{\widetilde{H}}_i$, $\underline{\widetilde{H}}^i$ the corresponding sheaves associated with the fiber bundle $\widetilde{p}$ we obtain sheaf epimorphisms $\underline{\widetilde{H}}_i \to \underline{H}_i$ and sheaf monomorphisms $\underline{H}^i \to \underline{\widetilde{H}}^i$ defined by the morphism sequences $(\alpha_{b*})_{b \in B}$, $(\alpha_b^*)_{b \in B}$.

We denote by f a generator of $\mathrm{Aut}_{\mathcal{M}}(\widetilde{\mathcal{M}})$ and by f_b the induced generator of $\mathrm{Aut}_{M_b}(\widetilde{M}_b)$. With these notations we obtain, for any $b \in B$, a commutative diagram with horizontal short exact sequences:

$$\begin{array}{ccccccccc} 0 \to & (\mathrm{id} - f_b)H_2(\widetilde{M}_b, \mathbb{Z}) & \hookrightarrow & H_2(\widetilde{M}_b, \mathbb{Z}) & \xrightarrow{\alpha_{b,*}} & H_2(M_b, \mathbb{Z}) & \to 0 \\ & \uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq & \\ 0 \to & (\mathrm{id} - f)H^0(B, \underline{\widetilde{H}}_2) & \hookrightarrow & H^0(B, \underline{\widetilde{H}}_2) & \xrightarrow{\alpha_*} & H^0(B, \underline{H}_2) & \to 0 \end{array} \ .$$

Let X be a class VII surface with topological type $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$ and Σ a strictly pseudo-convex non-separating embedded 3-sphere in X. A lift $\widetilde{\Sigma}$ of Σ to $\widetilde{X}$ separates $\widetilde{X}$ in two manifolds with common boundary $\widetilde{\Sigma}$, so we can define the pseudo-convex

and pseudo-concave ends of $\widetilde{X}$ as in the case of GSS surfaces (see section 1.2): the pseudo-convex end of $\widetilde{X}$ is the end defined by the connected component of $\widetilde{X}\setminus\widetilde{\Sigma}$ whose closure has strictly pseudo-concave boundary. One can prove that this labeling of the ends is coherent, i.e. independent of the choice of Σ and $\widetilde{\Sigma}$.

Theorem 3.1. *(first extension theorem) Let $p:\mathcal{X}\to B$ be a holomorphic family of class VII surfaces with 2-connected basis B, such that every fiber X_b*

1. *has the topological type $(S^1\times S^3)\#n\bar{\mathbb{P}}^2$,*
2. *admits a non-separating strictly pseudo-convex embedded 3-sphere Σ_b.*

Let $e=(e_b)_{b\in B}\in H^0(B,\underline{H}_2)$ with $e^2=\langle e,k\rangle=-1$ such that $B_e\neq\emptyset$. Let $\alpha:\widetilde{\mathcal{X}}\to\mathcal{X}$ be a universal cover of $\mathcal{X}$, and $\widetilde{e}=(\widetilde{e}_b)_{b\in B}$ a lift of e in $H^0(B,\underline{\widetilde{H}}_2)$. Then there exists an effective divisor $\widetilde{\mathcal{E}}\subset\widetilde{\mathcal{X}}$ with the following properties:

1. *For any point $b\in B_e$, the fiber $\widetilde{E}_b:=\widetilde{\mathcal{E}}\cap\widetilde{X}_b$ is a lift representing $\widetilde{e}_b$ of the unique simply exceptional divisor E_b in the class e_b,*
2. *For any point $b\in H_e$, the fiber $\widetilde{E}_b$ is bounded towards the pseudo-convex end of $\widetilde{X}_b$, but unbounded towards its pseudo-concave end.*
3. *$\widetilde{\mathcal{E}}$ is flat over B.*

Proof. Using Theorem 2.9 we get an effective divisor $\mathcal{E}_0\subset p^{-1}(B_e)$, flat over B_e, whose fiber over a point $b\in B_e$ is the unique simply exceptional divisor $E_b\subset X_b$ in the class e_b. Since E_b is a tree of rational curves (so simply connected), it can be lifted to $\widetilde{X}_b$, and the set of lifts can be identified with the set $\alpha_{b,*}^{-1}(e_b)\subset H_2(\widetilde{X}_b,\mathbb{Z})$. Moreover, since E_b has a simply connected neighborhood in $\mathcal{X}$, such a lift can be locally chosen such that it depends holomorphically on $b\in B_e$.

For $b\in B_e$ let $\widetilde{E}_b$ be the lift of E_b representing the given class $\widetilde{e}_b\in H_2(\widetilde{X}_b,\mathbb{Z})$. Using Proposition 2.3 we see that map $B_e\ni b\mapsto\widetilde{E}_b\in\mathcal{D}ou^e(\mathcal{X})$ is biholomorphic. The union

$$\widetilde{\mathcal{E}}_0:=\bigcup_{b\in B_e}\widetilde{E}_b$$

is an effective divisor of $\widetilde{p}^{-1}(B_e)\subset\widetilde{\mathcal{X}}$, and the restriction $\alpha_{|\widetilde{\mathcal{E}}_0}:\widetilde{\mathcal{E}}_0\to\mathcal{E}_0$ is biholomorphic.

The idea of the proof is to prove, using the Remmert-Stein theorem, that the closure $\widetilde{\mathcal{E}}:=\bar{\widetilde{\mathcal{E}}}_0$ of $\widetilde{\mathcal{E}}_0$ in $\widetilde{\mathcal{X}}$ is a divisor, and that this divisor has the desired properties. The irreducible components of $\widetilde{H}_e:=\widetilde{\mathcal{X}}\setminus\widetilde{p}^{-1}(B_e)=\widetilde{p}^{-1}(H_e)$ have the form $\widetilde{p}^{-1}(H)$, where H is an irreducible component of the divisor H_e (see Theorem 2.9). Let H_0 be such a component and $b_0\in H_0$. Using the second property of the fibers, we get an embedding $s_0:S^3\to X_{b_0}$ whose image Σ_0 is non-separating and strictly pseudo-convex. The image $\widetilde{\Sigma}_0:=\mathrm{im}(\widetilde{s}_0)$ of a lift $\widetilde{s}_0:S^3\to\widetilde{X}_{b_0}$ of this embedding to $\widetilde{X}_{b_0}$ separates $\widetilde{X}_{b_0}$ in two manifolds $\widetilde{X}_{b_0}^{\pm}$, the first with strictly pseudo-concave, the second with strictly pseudo-convex boundary $\widetilde{\Sigma}_0$.

We can deform the lift $\widetilde{s}_0$ to get a smooth family of embeddings $(\widetilde{s}_b:S^3\to\widetilde{\mathcal{X}})_{b\in U}$, where U is a connected open neighborhood of b_0 in B, such that $\widetilde{s}_{b_0}=\widetilde{s}_0$, and for every $b\in U$ the image $\widetilde{\Sigma}_b:=\mathrm{im}(\widetilde{s}_b)$ is a strictly pseudo-convex separating hypersurface of $\widetilde{X}_b$. Therefore, for every $b\in U$ the universal cover $\widetilde{X}_b$ of X_b decomposes as the union of two manifolds $\widetilde{X}_b^{\pm}$, the first with pseudo-concave, the second with pseudo-convex boundary $\widetilde{\Sigma}_b$. Similarly, $\widetilde{\mathcal{X}}_U:=\widetilde{p}^{-1}(U)$ decomposes as the union of two manifolds $\widetilde{\mathcal{X}}_U^{\pm}=\cup_{b\in U}\widetilde{X}_b^{\pm}$ with common boundary $\widetilde{\mathfrak{S}}_U:=\cup_{b\in U}\widetilde{\Sigma}_b$. We choose the generator f of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ such that the induced automorphism f_b of $\widetilde{X}_b$ maps $\widetilde{X}_b^-$ into itself (so it moves the points towards the pseudo-concave end) for every $b\in U$.

Choose a point $v_0 \in B_e \cap U$. Since the divisor $\widetilde{E}_{v_0} \subset \widetilde{X}_{v_0}$ is compact, we can find $m \in \mathbb{N}$ sufficiently large such that $(f_{v_0}^{-m}(\widetilde{X}_{v_0}^+)) \cap \widetilde{E}_{v_0} = \emptyset$. We claim

Claim: $(f^{-m}(\widetilde{\mathcal{X}}_U^+)) \cap \widetilde{\mathcal{E}}_0 = \emptyset$.

Indeed, let $V \subset U \cap B_e$ the subset of points v for which $(f_v^{-m}(\widetilde{X}_v^+)) \cap \widetilde{E}_v = \emptyset$. Since all divisors $\widetilde{E}_b$, $b \in B_e$ are compact, it follows easily that V is open in $U \cap B_e$. We will show that it is also closed in $U \cap B_e$; in order to check this, let $(v_n)_{n\in\mathbb{N}^*}$ be a sequence in V converging to a point $w \in U \cap B_e$. Since $\widetilde{E}_{v_n} \subset f_{v_n}^{-m}(\widetilde{X}_{v_n}^-)$ for every $n \in \mathbb{N}^*$ it follows that $\widetilde{E}_w \subset f_w^{-m}(\widetilde{X}_w^-)$. The intersection $\widetilde{E}_w \cap f_w^{-m}(\widetilde{\Sigma}_w)$ is empty, because $f_w^{-m}(\widetilde{\Sigma}_w)$ regarded as boundary of $f_w^{-m}(\widetilde{X}_w^-)$ is strictly pseudo-convex. Therefore $\widetilde{E}_w \cap f_w^{-m}(\widetilde{X}_w^+) = \emptyset$, so $w \in V$.

Therefore V is both open and closed in $U \cap B_e = U \setminus H_e$, which is connected, so $V = U \cap B_e$. Therefore $(f_u^{-m}(\widetilde{X}_u^+)) \cap \widetilde{\mathcal{E}}_0 = \emptyset$ for every $u \in U \cap B_e$, so for every $u \in U$; taking into account that $\bigcup_{u\in U} f_u^{-m}(\widetilde{X}_u^+) = f^{-m}(\widetilde{\mathcal{X}}_U^+)$, this proves the claim.

Note now that $f^{-m}(\widetilde{\mathcal{X}}_U^+)$ is a neighborhood of any point in $f_{b_0}^{-m-1}(\widetilde{X}_{b_0}^+)$. Therefore the claim implies that $\widetilde{H}_0$ contains points which do not belong to the closure $\bar{\widetilde{\mathcal{E}}}_0$ of the divisor $\widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{X}} \setminus \widetilde{H}_e$. Since this holds for every irreducible component $\widetilde{H}_0$ of $\widetilde{H}_e$, the existence of a divisor $\mathcal{E}$ satisfying the first property stated in the theorem follows now from the Remmert-Stein theorem (see [13] Korollar zu Satz 12, p. 300). For the second property note first that for any $b_0 \in H_e$ we have $(f_{b_0}^{-m}(\widetilde{X}_{b_0}^+)) \cap \widetilde{E}_{b_0} = \emptyset$, so $\widetilde{E}_{b_0}$ is indeed bounded towards the pseudo-convex end of $\widetilde{X}_{b_0}$. On the other hand $\widetilde{E}_{b_0}$ cannot be compact because, if it were, its projection on X_{b_0} would be a limit of simply exceptional divisors, so itself simply exceptional (contradicting $b_0 \in H_e$).

For the third property note that for every $b \in B$ the fiber $\widetilde{E}_b$ of $\widetilde{\mathcal{E}}$ is a divisor of $\widetilde{X}_b$, so the result follows from Lemma 2.4.

□

Remark 3.2. *Note that for $b \in B_e$ the divisor $\widetilde{E}_b$ represents the class $\widetilde{e}_b \in H_2(\widetilde{X}_b, \mathbb{Z})$. This property cannot be extended for $b \in H_e$ because in this case $\widetilde{E}_b$ is no longer compact. However, using the flatness property of $\widetilde{\mathcal{E}}$ over B we see that for every $b \in B$*

1. *$c_1(\mathcal{O}_{X_b}(\widetilde{E}_b))$ is the image of the Poincaré dual $PD(\widetilde{e}_b) \in H_c^2(\widetilde{X}_b, \mathbb{Z})$ of $\widetilde{e}_b$ via the natural morphism $H_c^2(\widetilde{X}_b, \mathbb{Z}) \to H^2(\widetilde{X}_b, \mathbb{Z})$.*
2. *$\widetilde{E}_b$ represents the image of $\widetilde{e}_b$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(\widetilde{X}_b, \mathbb{Z})$.*

Taking into account Remark 3.2 and Theorem 3.1 we obtain immediately Theorem 1.2 stated in the introduction.

Remark 3.3. *The assignment $(p : \mathcal{X} \to B, \alpha : \widetilde{\mathcal{X}} \to \mathcal{X}, \widetilde{e}) \mapsto \widetilde{\mathcal{E}} \subset \widetilde{\mathcal{X}}$ given by the proof of Theorem 3.1 is compatible with base changes $\chi : B' \to B$ satisfying $\chi^{-1}(B_e) \neq \emptyset$.*

Using this remark one can prove that the assumption $B_e \neq \emptyset$ in Theorem 3.1 can be replaced with one of the following equivalent assumptions:

(i) *For every $b \in B$ there exists small deformations of X_b admitting a simply exceptional divisors in the class e_b.*
(ii) *For every $b \in B$ the versal deformation of X_b contains surfaces admitting a simply exceptional divisors in the class e_b.*

In particular, the theorem holds when all the fibers are GSS surfaces, because any GSS surface can be deformed into simply blown up primary Hopf surfaces. We will see in the next section that for families of GSS surfaces, the divisors $\widetilde{E}_b$, $b \in H_e$ are always given by infinite series of compact curves.

Using Remark 3.3 we obtain Theorem 1.3 stated in the introduction.

Our second extension theorem concerns families of class VII surfaces for which the existence of strictly pseudo-convex embedded 3-spheres is assumed only for the fibers X_b, $b \in B \setminus A$, where A is an analytic subset of codimension ≥ 2 of B and contained in the divisor H_e. This theorem contains obviousy Theorem 1.4 stated in the introduction as a special case.

The ends of the universal covers $\widetilde{X}_b$ of the fibers form a trivial double cover of B, so one can speak about the pseudo-convex and the pseudo-concave end of X_b even for points $b \in A$.

Theorem 3.4. *(second extension theorem) Let $p : \mathcal{X} \to B$ be a holomorphic family of class VII surfaces with 2-connected basis B. Let $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2) = H_2(\mathcal{X}, \mathbb{Z})$ with $e^2 = \langle e, k\rangle = -1$ such that $B_e \neq \emptyset$, and A an analytic subset of codimension ≥ 2 of B such that $A \subset H_e$ and, for every $b \in H_e \setminus A$, the fiber X_b has the properties*

(1) *has the topological type $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$,*
(2) *admits a non-separating strictly pseudo-convex embedded 3-sphere Σ_b.*

Let $\alpha : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a universal cover of $\mathcal{X}$, and $\widetilde{e} = (\widetilde{e}_b)_{b\in B}$ a lift of e in $H^0(B, \underline{\widetilde{H}}_2)$. Then there exists an effective divisor $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{X}}$ with the following properties:

(i) *For any point $b \in B_e$, the fiber $\widetilde{E}_b := \widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is a lift representing $\widetilde{e}_b$ of the unique simply exceptional divisor E_b in the class e_b.*
(ii) *for any point $b \in H_e = B \setminus B_e$, the fiber $\widetilde{E}_b$ is a divisor of $\widetilde{X}_b$ which is bounded towards the pseudo-convex end of $\widetilde{X}_b$, but unbounded towards its pseudo-concave end.*
(iii) *$\widetilde{\mathcal{E}}$ is flat over B.*

Proof. Applying our first extension theorem Theorem 3.1 to the family $\widetilde{p}^{-1}(B \setminus A) \to B \setminus A$ we get an effective divisor $\widetilde{\mathcal{E}}^0 \subset \widetilde{p}^{-1}(B \setminus A)$ flat over $B \setminus A$ satisfying the property (i) and also the property (ii) for any point $b \in B \setminus A$. So $\widetilde{\mathcal{E}}^0$ is the closure in $\widetilde{p}^{-1}(B \setminus A)$ of the lift $\widetilde{\mathcal{E}}_0 \subset \widetilde{p}^{-1}(B \setminus H_e)$ of $\mathcal{E}_0$ obtained by lifting fiberwise the simply exceptional divisors E_b in the homology class $\widetilde{e}_b$ (for $b \in B_e$).

Since $\widetilde{p}^{-1}(A)$ has codimension ≥ 2 in $\widetilde{\mathcal{X}}$, the closure $\widetilde{\mathcal{E}} := \overline{\widetilde{\mathcal{E}}^0}$ of $\widetilde{\mathcal{E}}^0$ in $\widetilde{\mathcal{X}}$ is an effective divisor of $\widetilde{\mathcal{X}}$ by the second Remmert-Stein extension theorem (see [13] Satz 13 p. 299), and with this choice the first statement of the theorem is proved.

For the second statement, the first problem is to show that for $b \in A$ the intersection $\widetilde{E}_b := \widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is a divisor of $\widetilde{X}_b$, i.e. that $\widetilde{E}_b$ does not coincide with the whole surface $\widetilde{X}_b$. We will show first that $\widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is bounded towards the pseudo-convex end, which will imply that $\widetilde{E}_b \cap \widetilde{X}_b \neq X_b$, so this intersection is indeed a divisor of X_b with the desired property.

Consider the semigroup $G_+ := \{f^n |\ n \in \mathbb{N}\}$ of transformations of $\widetilde{\mathcal{X}}$ which act fiberwise with respect to the fibration $\widetilde{p} : \widetilde{\mathcal{X}} \to B$. Here we choose the generator f of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ as in the proof of Theorem 3.1, so that it moves the points towards the pseudo-concave ends of the fibers $\widetilde{X}_b$. Using the same arguments as in the proof of Theorem 3.1 we see that the closure $\widetilde{\mathcal{F}}^0$ of the effective divisor

$$\widetilde{\mathcal{F}}_0 := G_+(\widetilde{\mathcal{E}}_0) = \bigcup_{n\in\mathbb{N}} f^n(\widetilde{\mathcal{E}}_0)$$

in $\widetilde{p}^{-1}(B \setminus A)$ is also an effective divisor of $\widetilde{p}^{-1}(B \setminus A)$, flat over $B \setminus A$. Using the second Remmert-Stein extension theorem cited above we get two effective divisors

$$\widetilde{\mathcal{E}} = \overline{\widetilde{\mathcal{E}}^0} = \bar{\widetilde{\mathcal{E}}}_0\ ,\ \widetilde{\mathcal{F}} = \overline{\widetilde{\mathcal{F}}^0} = \bar{\widetilde{\mathcal{F}}}_0$$

in $\widetilde{\mathcal{X}}$ (where $^-$ denotes everywhere closure in $\widetilde{\mathcal{X}}$). For every $n \in \mathbb{N}$, we have $f^n(\widetilde{\mathcal{E}}_0) \subset \widetilde{\mathcal{F}}$ so, since the right hand space is closed, we get $f^n(\widetilde{\mathcal{E}}) \subset \widetilde{\mathcal{F}}$, which proves the inclusion

$$\forall n \in \mathbb{N}\ , f^n(\widetilde{\mathcal{E}}) \subset \widetilde{\mathcal{F}}. \tag{4}$$

Recall that the effective divisor $\widetilde{\mathcal{E}}_0$ of $\widetilde{p}^{-1}(B_e)$ is identified with $\mathcal{E}_0$ via the covering map α. Since the set of irreducible components of $\mathcal{E}_0$ is finite (bounded by the minimal number of irreducible components of the simply exceptional divisors E_b as b varies in B_e), the same will hold for $\widetilde{\mathcal{E}}_0$. Let $\widetilde{\mathcal{C}}_0$ be an irreducible component of $\widetilde{\mathcal{E}}_0$. Using the two Remmert-Stein theorems cited above, we see that the closure $\widetilde{\mathcal{C}}^0$ of $\widetilde{\mathcal{C}}_0$ in $\widetilde{p}^{-1}(B_e)$ is an effective divisor in this manifold, and that $\widetilde{\mathcal{C}} := \widetilde{\mathcal{C}}^0 = \widetilde{\mathcal{C}}_0$ is an effective divisor of $\widetilde{\mathcal{X}}$. We will show that

Claim 1: $\widetilde{\mathcal{C}} \cap \widetilde{X}_b$ is bounded towards the pseudo-convex end of $\widetilde{X}_b$ for every $b \in A$.

Since $\widetilde{\mathcal{E}}$ is the (finite!) union of the closures $\widetilde{\mathcal{C}}$ of the irreducible components $\widetilde{\mathcal{C}}_0$ of $\widetilde{\mathcal{E}}_0$, the claim implies obviously the second statement of the theorem.

We prove now Claim 1. Choose a point $b_0 \in A$. Since p is a locally trivial differentiable fiber bundle, it follows that there exists a smooth embedding $s_0 : S^3 \to X_{b_0}$ whose image Σ_0 does not disconnect X, and let $\widetilde{s}_0 : S^3 \to \widetilde{X}_{b_0}$ be a lift of s_0 to $\widetilde{X}_{b_0}$. We can deform $\widetilde{s}_0$ to get a smooth family of embeddings $(\widetilde{s}_b : S^3 \to \widetilde{\mathcal{X}})_{b\in U}$, where U is a connected open neighborhood of b_0 in B, such that $\widetilde{s}_{b_0} = \widetilde{s}_0$ and s_b is an embedding of S^3 in $\widetilde{X}_b$ for every $b \in U$. We can regard this family as an embedding $\widetilde{s} : U \times S^3 \to \widetilde{\mathcal{X}}_U := \widetilde{p}^{-1}(U)$, whose image $\widetilde{\Sigma}_U$ separates $\widetilde{\mathcal{X}}_U$ in two connected components. Moreover, $\widetilde{\mathcal{X}}_U$ can be written as an infinite union

$$\widetilde{\mathcal{X}}_U = \bigcup_{n\in\mathbb{Z}} \widetilde{\mathcal{X}}_U^n\ ,$$

where $\widetilde{\mathcal{X}}_U^n = f^n(\widetilde{\mathcal{X}}_U^0)$ is a manifold with two boundary components $f^n(\widetilde{\Sigma}_U)$, $f^{n+1}(\widetilde{\Sigma}_U)$.

Let $V \Subset U$ an open neighborhood of b_0 with compact closure $\bar{V} \subset U$, and put

$$\widetilde{\mathcal{X}}_{\bar{V}} := \widetilde{p}^{-1}(\bar{V})\ ,\ \widetilde{\Sigma}_{\bar{V}} := \widetilde{\Sigma}_U \cap \widetilde{\mathcal{X}}_{\bar{V}}\ ,\ \widetilde{\mathcal{X}}_{\bar{V}}^n := \widetilde{\mathcal{X}}_U^n \cap \widetilde{\mathcal{X}}_{\bar{V}}\ ,$$

and note that the spaces $\widetilde{\mathcal{X}}_{\bar{V}}^n = f^n(\widetilde{\mathcal{X}}_{\bar{V}}^0)$ are all compact. Our claim follows from

Claim 2: $\widetilde{\mathcal{C}} \cap \widetilde{\mathcal{X}}_{\bar{V}}$ is bounded towards the pseudo-convex end.

Indeed if, by reductio ad absurdum, $\widetilde{\mathcal{C}} \cap \widetilde{\mathcal{X}}_{\bar{V}}$ were not bounded towards the pseudo-convex end, then the same would be true for $f^n(\widetilde{\mathcal{C}}) \cap \widetilde{\mathcal{X}}_{\bar{V}}$ for every $n \geq 0$. Therefore the irreducible divisors $f^n(\widetilde{\mathcal{C}})$ intersect the compact subset $\widetilde{\mathcal{X}}_{\bar{V}}^0$ of $\widetilde{\mathcal{X}}$ for all sufficiently large $n \geq 0$. Using Lemma 3.5 below, it would follow that the sequence $(f^n(\widetilde{\mathcal{C}}))_{n\geq 0}$ is finite, which is of course impossible, because the intersections of these divisors with $\widetilde{p}^{-1}(B_e)$ are pairwise different.

The third statement of the theorem follows now using the same arguments as in the proof of the similar statement in Theorem 3.1. □

Lemma 3.5. *Let M be a connected complex manifold, and let $(D_n)_{n\in\mathbb{N}}$ be a sequence of effective irreducible divisors in M. Suppose that*

1. *The union $D := \bigcup_{n\in\mathbb{N}} D_n$ is a divisor of M, and*
2. *there exists a compact subspace $K \subset M$ such that $K \cap D_n \neq \emptyset$ for every $n \in \mathbb{N}$.*

Then the set of divisors $\{D_n |\ n \in \mathbb{N}\}$ is finite.

Proof. If not, there would exist a subsequence $(D_{n_m})_{m\in\mathbb{N}}$ of $(D_n)_{n\in\mathbb{N}}$ such that $D_{n_k} \neq D_{n_l}$ for every $k \neq l$. Since all these divisors are irreducible, it follows that:

Remark: For $k \neq l$, the analytic set $D_{n_k} \cap D_{n_l}$ does not contain any codimension 1 irreducible component.

For every $m \in \mathbb{N}$, let $x_m \in K \cap D_{n_m}$, and let $x \in K$ be the limit of a convergent subsequence $(x_{m_s})_{s\in\mathbb{N}}$ of $(x_m)_{m\in\mathbb{N}}$. Let U be a connected open neighborhood of x, so U contains all the points x_{m_s} with $s \geq s_0$ for a sufficiently large index $s_0 \in \mathbb{N}$. For every $s \geq s_0$ we consider an irreducible component E_s of the intersection $D_{n_{m_s}} \cap U$, so E_s is a *non-empty* effective divisor of U. Taking into account the remark above, it follows that $E_s \neq E_t$ for $s \neq t$.

Therefore, the intersection $D \cap U = \left(\bigcup_{n\in\mathbb{N}} D_n\right) \cap U$ contains the infinite union $\bigcup_{s\geq s_0} E_s$ of pairwise distinct irreducible effective divisors E_s, so $D \cap U$ cannot be a divisor of U. This contradicts the first assumption of our hypothesis.

□

4. The case of GSS surfaces. The fundamental series

We begin with the following

Definition 4.1. *Let X be a compact complex surface. We say that X contains a global spherical shell (GSS), or that X is a GSS surface, if there exists a biholomorphic map $\varphi : U \to X$ from a neighborhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere S^3 into X such that $X \setminus \varphi(S^3)$ is connected.*

Primary Hopf surfaces are the simplest examples of GSS surfaces. The differential topological type of a GSS surface X with $b_2(X) = n$ is $(S^1 \times S^3)\#n\overline{\mathbb{P}^2}$, so $H_2(X,\mathbb{Z})$ admits a (unordered) basis $\mathfrak{B} \subset H_2(X,\mathbb{Z})$ trivializing the intersection form q_X of X, i.e. such that

$$e' \cdot e'' = \begin{cases} 0 & \text{if} \quad e' \neq e'' \\ -1 & \text{if} \quad e' = e'' \end{cases} .$$

Decomposing the Chern class $c_1(\mathcal{K}_X) = -c_1(X) \in H^2(X,\mathbb{Z})$ with respect to the Poincaré dual basis $\mathfrak{B}^\vee = \{PD(e)|\ e \in \mathfrak{B}\}$ of $H^2(X,\mathbb{Z})$ we get

$$c_1(\mathcal{K}_X) = \sum_{e\in\mathfrak{B}} k_e PD(e)$$

with $k_e \equiv 1 \bmod 2$ (because $c_1(\mathcal{K}_X)$ is a characteristic class) and $\sum_{e\in\mathfrak{B}}^{n-1} k_e^2 = -c_1(X)^2 = c_2(X) = n$. This shows that $k_e \in \{\pm 1\}$ so, replacing some of the elements $e \in \mathfrak{B}$ by $-e$ if necessary, we may assume that

$$c_1(\mathcal{K}_X) = \sum_{e\in\mathfrak{B}} PD(e) \ , \tag{5}$$

or equivalently

$$\forall e \in \mathfrak{B} \ , \ \langle e, c_1(\mathcal{K}_X)\rangle = -1 \ . \tag{6}$$

There exists a unique unordered q_X-trivializing basis $\mathfrak{B}$ with this property, and this basis will be called *the standard basis* of $H_2(X,\mathbb{Z})$. Note that the homology class of any simply exceptional divisor of X (if such a divisor exists) is an element of its standard basis.

For surfaces X with GSS, the second Betti number $b_2(X)$ is equal to the number of rational curves in X. A *marked GSS surface* is a pair (X, C_0) consisting of GSS surface with $b_2(X) \geq 1$ and a rational curve C_0 in X.

Any minimal marked GSS surface (X, C_0) with $n = b_2(X) \geq 1$ can be obtained using a simple 2-step construction consisting of an iterated blow up followed by a holomorphic surgery. More precisely, let $\Pi = \Pi_0 \circ \cdots \circ \Pi_{n-1} : B^\Pi \to B$ be a sequence of n blowing ups such that

- the first blowing up $\Pi_0 : B^{\Pi_0} \to B$ blows up the origin $O_{-1} := (0,0)$ in the 2-dimensional unit ball $B \subset \mathbb{C}^2$,
- for $i = 0, \ldots, n-2$ the blowing up $\Pi_{i+1} : B^{\Pi_0 \circ \cdots \circ \Pi_{i+1}} \to B^{\Pi_0 \circ \cdots \circ \Pi_i}$ blows up a point $O_i \in \Pi_i^{-1}(O_{i-1})$ in the surface $B^{\Pi_0 \circ \cdots \circ \Pi_i}$ obtained at the previous step.

Applying the same sequence of blowing ups to a ball $B(r)$ of radius r and to its closure $\bar{B}(r)$ we obtain complex surfaces (respectively compact complex surfaces with boundary) which will be denoted by $B(r)^\Pi$, respectively $\bar{B}(r)^\Pi$.

Let now $\sigma : \bar{B} \to B^\Pi$ an embedding which extends to a biholomorphism from a neighborhood of $\bar{B}$ onto a small open ball in B^Πsuch that $\sigma(0) = O_{n-1} \in \Pi_{n-1}^{-1}(O_{n-2})$. One can associate to the pair (Π, σ) a minimal surface $X = X(\Pi, \sigma)$ by removing from $\bar{B}^\Pi$ the image $\sigma(B)$, and identifying the two boundary components $\partial\bar{B}^\Pi$, $\sigma(\partial\bar{B})$ of the compact manifold with boundary $A := \bar{B}^\Pi \setminus \sigma(B)$ by $\sigma \circ \Pi$, the holomorphic structure on the resulting differentiable manifold being defined such that the natural surjective locally diffeomorphic map

$$q_{\Pi,\sigma} : B(1+\epsilon)^\Pi \setminus \sigma(\bar{B}(1-\epsilon)) \longrightarrow X(\Pi, \sigma) \ .$$

is a biholomorphism. One can prove that for any marked surface (X, C_0) there exists a pair (Π, σ) and a biholomorphism $X \xrightarrow{\simeq} X(\Pi, \sigma)$ such that C_0 corresponds to the image via $q_{\Pi,\sigma}$ of the first exceptional curve $\Pi_0^{-1}(O_{-1})$. We will identify X with $X(\Pi, \sigma)$ via this biholomorphism.

For any $i \in \{0, \ldots, n-2\}$ consider a small closed ball $\bar{B}_i$ centered at $O_i \in B^{\Pi_0 \circ \cdots \circ \Pi_i}$ and denote by S_i the lift of its boundary to B^Π. It is also convenient to denote by $S_{-1} := \partial\bar{B}^\Pi$, $\bar{B}_{n-1} := \sigma(\bar{B})$, and $S_{n-1} := \partial\bar{B}_{n-1} = \sigma(\partial\bar{B})$. In this way A decomposes as the union

$$A = \bigcup_{i=0}^{n-1} \mathfrak{A}_i \ ,$$

where each $\mathfrak{A}_i$ is a compact surface with two boundary components: a strictly pseudo-convex component $\partial_-(\mathfrak{A}_i) \simeq S_{i-1}$ and the strictly pseudo-concave component $\partial_+(\mathfrak{A}_i) \simeq S_i$. Every $\mathfrak{A}_i$ is biholomorphic to the manifold obtained from a blown up closed ball by removing a small open ball centered at a point of the exceptional divisor.

The universal covering space $\widetilde{X}$ can be obtained as an infinite union of copies A_k of A, the pseudo-convex boundary component of A_k being identified with the pseudo-concave boundary component of the previous annulus A_{k-1} via $\sigma \circ \Pi$. We will identify A_0 with A, so A_0 decomposes as $\bigcup_{i=0}^{n-1} \mathfrak{A}_i$; correspondingly we get decompositions

$$\forall j \in \mathbb{Z} \ , \ A_j = \bigcup_{i=0}^{n-1} \mathfrak{A}_{nj+i} \ , \ \widetilde{X} = \bigcup_{s \in \mathbb{Z}} \mathfrak{A}_s \ ,$$

where $\mathfrak{A}_{nj+i}$ is a copy of $\mathfrak{A}_i$ for every $i \in \{0, \ldots, n-1\}$, and $\widetilde{X}$ is obtained by identifying the pseudo-convex boundary component $\partial_-(\mathfrak{A}_s)$ of any $\mathfrak{A}_s$ with the pseudo-concave boundary component $\partial_+(\mathfrak{A}_{s-1})$ of the previous piece, in the obvious way. For every $k \in \mathbb{Z}$ we put

$$\widetilde{X}_k := \bigcup_{s \leq k} \mathfrak{A}_s \ .$$

Note that $\widetilde{X}_{nj+n-1} = \bigcup_{j' \leq j} A_{j'}$. The boundary $\partial\widetilde{X}_k$ is the strictly pseudo-concave 3-sphere $\partial_+(\mathfrak{A}_k) \simeq \partial_+(\mathfrak{A}_{k-n\left[\frac{k}{n}\right]}) \simeq S_{k-n\left[\frac{k}{n}\right]}$. We denote by $\widehat{X}_k$ the surface obtained

by gluing (a copy of) $\bar{B}_{k-n\left[\frac{k}{n}\right]}$ along this boundary. For every $k<l$ we have obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{X}_k & \xrightarrow{\iota_{k,l}} & \widetilde{X}_l \\ \iota_k \downarrow & & \downarrow \iota_l \\ \widehat{X}_k & \xleftarrow{\pi_{k,l}} & \widehat{X}_l \end{array} ,$$

where ι_k, ι_l, $\iota_{k,l}$ are the obvious inclusions, and $\pi_{k,l}$ is an order $l-k$ iterated blow up at the center of the ball $\bar{B}_{k-n\left[\frac{k}{n}\right]}$. For $k<l<m$ one has obviously $\iota_{l,m}\circ\iota_{k,l}=\iota_{k,m}$, $\pi_{k,l}\circ\pi_{l,m}=\pi_{k,m}$. Note that, for $i\in\{1,\dots,n-1\}$ the blow up $\pi_{nj+i-1,nj+i}$ corresponds to Π_i via obvious identifications, whereas $\pi_{nj-1,nj}$ corresponds to $\Pi_0\circ\sigma^{-1}$. We will denote by E_k the exceptional curve of the blow up $\pi_{k-1,k}:\widehat{X}_k\to\widehat{X}_{k-1}$. The pre-image $\iota_k^{-1}(E_k)$ is of course non-compact, but its Zariski closure in $\widetilde{X}$ is a compact rational curve $\widetilde{C}_k$ which can be explicitly obtained in the following way (see [4] for details):

Remark 4.2. *There exists $s\in\{1,\dots,n+1\}$ such that the proper transform E_k^s of E_k in $\widehat{X}_{k+s}$ is contained in the image of ι_{k+s}, so $\widetilde{C}_k:=\iota_{k+s}^{-1}(E_k^s)$ is a compact rational curve in $\widetilde{X}$, which can be identified with the Zariski closure of $\iota_k^{-1}(E_k)$.*

Note now that the group morphisms $H_2(\iota_k):H_2(\widetilde{X}_k,\mathbb{Z})\to H_2(\widehat{X}_k,\mathbb{Z})$ are isomorphisms, and that via the isomorphisms $H_2(\iota_k)$, $H_2(\iota_{k-1})$, the monomorphism $H_2(\iota_{k-1,k})$ defines a right splitting of the short exact sequence

$$0\to\mathbb{Z}[E_k]\to H_2(\widehat{X}_k,\mathbb{Z})\xrightarrow{H_2(\pi_{k-1,k})}H_2(\widehat{X}_{k-1},\mathbb{Z})\to 0\ .$$

We denote by $\widetilde{e}_k$ the image in $H_2(\widetilde{X}_k,\mathbb{Z})$ of the class $[E_k]$ via $H_2(\iota_k)^{-1}$; to save on notations we will use the same symbol for the images of $\widetilde{e}_k$ in $H_2(\widetilde{X}_l,\mathbb{Z})$ $(l\geq k)$ and in $H_2(\widetilde{X},\mathbb{Z})$. With these conventions we can write

$$H_2(\widetilde{X}_k,\mathbb{Z})=\bigoplus_{s\leq k}\mathbb{Z}\widetilde{e}_s\ ,\ H_2(\widetilde{X},\mathbb{Z})=\bigoplus_{s\in\mathbb{Z}}\mathbb{Z}\widetilde{e}_s$$

Using Remark 4.2 we see immediately that, via this decomposition, the homology class $[\widetilde{C}_k]\in H_2(\widetilde{X},\mathbb{Z})$ decomposes as

$$[\widetilde{C}_k]=\widetilde{e}_k-\sum_{i=1}^{s_k}\widetilde{e}_{k+i}\text{ where }1\leq s_k\leq n+1\ . \tag{7}$$

Note that one has obviously

$$s_k=-\widetilde{C}_k^2-1\ . \tag{8}$$

The n compact curves of the minimal GSS surface $X=X(\Pi,\sigma)$ are just the projections $C_k:=\alpha(\widetilde{C}_{nj+k})$ of the compact curves we obtained in the universal cover $\widetilde{X}$. Note also that the set $\{e_k:=\alpha_*(\widetilde{e}_k)|\ k\in\{0,\dots,n-1\}\}$ is precisely the standard basis $\mathfrak{B}$ of X.

We will show now that any class $\widetilde{e}_i$ decomposes formally in a well defined way as an infinite series of classes of compact curves which is bounded towards the pseudo-convex end. These formal identities correspond to equalities in the Borel-More homology of $\widetilde{X}$. We will begin with several interesting examples:

Exemple 4.3. *Enoki surfaces.*

An Enoki surface X has a cycle $\sum_{i=0}^{n-1}C_i$ of $n=b_2(X)$ rational curves. We have

$$[C_i]=e_i-e_{i+1},\quad i\in\mathbb{Z}_n\ .$$

In the universal cover $\widetilde{X}$ we have curves $\widetilde{C}_i$ representing the 2-homology classes $[\widetilde{C}_i] = \widetilde{e}_i - \widetilde{e}_{i+1},\ i \in \mathbb{Z}$, therefore we get the decomposition

$$\widetilde{e}_i = \sum_{j=0}^{\infty} [\widetilde{C}_{i+j}] \ .$$

Exemple 4.4. *Minimal GSS surfaces X with $b_2(X) = 1$.*

There are two classes of minimal GSS surfaces X with $b_2(X) = 1$:

- Enoki surfaces with $b_2 = 1$,
- Inoue-Hirzebruch surfaces with one cycle,

The first class has been treated above, so we discuss the second one. The curves in the universal cover $\widetilde{X}$ of an Inoue-Hirzebruch surface X with $b_2(X) = 1$ form two chains of rational curves

$$\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i}, \quad \sum_{i\in\mathbb{Z}} \widetilde{C}_{2i+1}$$

with $[\widetilde{C}_i] = \widetilde{e}_i - (\widetilde{e}_{i+1} + \widetilde{e}_{i+2})$. Using the Fibonacci sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$u_0 = 1, u_1 = 1,\ u_n = u_{n-2} + u_{n-1}, n \geq 2$$

we get

$$\widetilde{e}_i = \sum_{n\geq 0} u_n \widetilde{C}_{i+n} \ .$$

Exemple 4.5. *Minimal GSS surfaces X with $b_2(X) = 2$.*

There are four classes of minimal GSS surfaces X with $b_2(X) = 2$.

- Enoki surfaces with $b_2 = 2$,
- Intermediate surfaces,
- Inoue-Hirzebruch surfaces with one cycle,
- Inoue-Hirzebruch surface with two cycles consisting of a rational curve with a double point.

The case of Enoki surfaces has been treated above for any $b_2 > 0$.

Intermediate surfaces. An intermediate surface with $b_2 = 2$ has a rational curve with a double point C_0 and a non-singular rational curve C_1 with

$$C_0^2 = -1, \quad C_1^2 = -2, \quad C_0 C_1 = 1 \ .$$

Using our conventions and notations we get decompositions

$$[C_0] = -e_1,\ [C_1] = e_1 - e_0 \ .$$

In the universal covering space $\widetilde{X}$ we have an infinite chain of curves $\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i}$ with pairwise disjoint trees $\widetilde{C}_{2i+1}$, $i \in \mathbb{Z}$ which decompose as follows:

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}\ ,\ [\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} \ .$$

We have then

$$\begin{aligned} \widetilde{e}_{2i} &= \sum_{j=0}^{\infty} 2^j \Big\{ (\widetilde{e}_{2(i+j)} - \widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2}) + (\widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2}) \Big\} \\ &= \sum_{j=0}^{\infty} 2^j \Big\{ \widetilde{C}_{2(i+j)} + \widetilde{C}_{2(i+j)+1} \Big\} \end{aligned}$$

$$\widetilde{e}_{2i+1} = (\widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}) + \widetilde{e}_{2i+2} = [\widetilde{C}_{2i+1}] + \sum_{j=0}^{\infty} 2^j \Big\{ [\widetilde{C}_{2(i+j+1)}] + [\widetilde{C}_{2(i+j+1)+1}] \Big\} \ .$$

Inoue-Hirzebruch surfaces with $b_2 = 2$ and one cycle. Such a surface has a cycle C_0+C_1 of non-singular rational curves, where

$$C_0^2 = -4\ ,\ C_1^2 = -2\ ,\ C_0C_1 = 2\ .$$

More precisely

$$[C_0] = -2e_1\ ,\ [C_1] = e_1 - e_0\ .$$

In the universal cover $\widetilde{X}$ the homology classes of the compact curves $\widetilde{C}_i$ decompose as

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} - \widetilde{e}_{2i+3}\ ,\ [\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}\ .$$

These curves form two disjoint infinite chains of rational curves

$$\sum_{i\in\mathbb{Z}} \widetilde{C}_{4i} + \widetilde{C}_{4i+3}\ ,\ \sum_{i\in\mathbb{Z}} \widetilde{C}_{4i+1} + \widetilde{C}_{4i+2}\ .$$

We define by induction the following sequences of positive integers $(a_j)_{j\in\mathbb{N}}$, $(b_j)_{j\in\mathbb{N}}$ where $a_0 = b_0 = 1$, and

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = A^j \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \text{ with } A := \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\ .$$

Then

$$\begin{aligned}
\widetilde{e}_{2i} &= \sum_{j=0}^{\infty} \Big\{ a_j \big(\widetilde{e}_{2(i+j)} - \widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2} - \widetilde{e}_{2(i+j)+3}\big) + b_j \big(\widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2}\big) \Big\} \\
&= \sum_{j=0}^{\infty} a_j[\widetilde{C}_{2(i+j)}] + b_j[\widetilde{C}_{2(i+j)+1}] \\
&= [\widetilde{C}_{2i}] + [\widetilde{C}_{2i+1}] + 2[\widetilde{C}_{2i+2}] + 3[\widetilde{C}_{2i+3}] + 5[\widetilde{C}_{2i+4}] + 7[\widetilde{C}_{2i+5}] + 12[\widetilde{C}_{2i+6}] + \cdots
\end{aligned}$$

$$\widetilde{e}_{2i+1} = (\widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}) + \widetilde{e}_{2i+2} = [\widetilde{C}_{2i+1}] + \sum_{j=0}^{\infty} a_j[\widetilde{C}_{2(i+j+1)}] + b_j[\widetilde{C}_{2(i+j+1)+1}]\ .$$

Inoue-Hirzebruch surfaces with $b_2 = 2$ and two cycles. In this case we have two cycles C_0, C_1 consisting of a rational curve with double point. The intersection numbers are

$$C_0^2 = -1, \quad C_1^2 = -1\ ,$$

and the decompositions of $[C_i]$ with respect to the standard basis are

$$[C_0] = -e_1, \quad [C_1] = -e_0\ .$$

In $\widetilde{X}$ we have two disjoint chains of rational curves $\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i}$, $\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i+1}$, where

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}\ ,[\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} - \widetilde{e}_{2i+3}\ .$$

Therefore denoting again by $(u_j)_{j\in\mathbb{N}}$ the Fibonacci sequence used before we obtain:

$$\begin{aligned}
\widetilde{e}_{2i} &= \sum_{j\geq 0} u_j(\widetilde{e}_{2i+j} - \widetilde{e}_{2i+j+1} - \widetilde{e}_{2i+j+2}) &&= \sum_{j\geq 0} u_j[\widetilde{C}_{2i+j}] \\
\widetilde{e}_{2i+1} &= \sum_{j\geq 0} u_j(\widetilde{e}_{2i+j+1} - \widetilde{e}_{2i+j+2} - \widetilde{e}_{2i+j+3}) &&= \sum_{j\geq 0} u_j[\widetilde{C}_{2i+j+1}]\ .
\end{aligned}$$

The next result deals with the general case:

Proposition 4.6. *Let $X = X(\Pi, \sigma)$ be a GSS surface with $b_2(X) = n \geq 1$ and $(\widetilde{e}_i)_{i\in\mathbb{Z}}$ be the standard base of $H_2(\widetilde{X}, \mathbb{Z})$ constructed above. Then for any $i \in \mathbb{Z}$ there exists a well defined infinite series with positive integer coefficients $\sum_{j\geq 0} \alpha_j^i[\widetilde{C}_{i+j}]$ whose sum is*

the image of $\widetilde{e}_i$ *in the Borel-Moore homology group* $H_2^{\mathrm{BM}}(\widetilde{X},\mathbb{Z})$. *Moreover, the sequence* $(\alpha^i_j)_{j\geq 0}$ *is always increasing, has* $\alpha^i_0=1$, *and*

$$\begin{cases} \forall j, \quad \alpha^i_j = 1 & iff \quad X \text{ is a Enoki surface} , \\ \lim_{j\to\infty}\alpha^i_j=\infty & iff \quad X \text{ is not a Enoki surface} . \end{cases}$$

Proof. We determine the coefficient α^i_k of $[\widetilde{C}_{i+k}]$ by induction on $k\geq 0$ such that we have the congruence

$$\widetilde{e}_i \equiv \sum_{j=0}^{k}\alpha^i_j[\widetilde{C}_{i+j}] \text{ modulo classes } \widetilde{e}_j \ , \ j>k \ . \tag{9}$$

For $k=0$ we use formula (7) to obtain

$$[C_i]=\widetilde{e}_i-\sum_{j=1}^{s_i}\widetilde{e}_{i+j}\equiv \widetilde{e}_i \text{ mod classes } \widetilde{e}_{i+j} \ , \ j>0 \ ,$$

so we get a well-defined solution $\alpha^i_0=1$.

Suppose now that the development is determined till $k\in\mathbb{N}$ so that

$$\widetilde{e}_i\equiv\sum_{j=0}^{k}\alpha^i_j[\widetilde{C}_{i+j}]=\sum_{j=0}^{k}\alpha^i_j\big(\widetilde{e}_{i+j}-\sum_{l=1}^{s_{i+j}}\widetilde{e}_{i+j+l}\big) \text{ mod classes } \widetilde{e}_{i+j} \ , \ j>k \ .$$

We define $J^i_{k+1}=\big\{j\in\{0,\dots,k\} \mid \ i+j+1\leq i+k+1\leq i+j+s_{i+j}\big\}$. With this notation we obtain a well defined solution

$$\alpha^i_{k+1}=\sum_{l\in J_{k+1}}\alpha^i_l \tag{10}$$

which guarantees the congruence (9) for $k+1$.

We prove now that the obtained sequence $(\alpha^i_j)_{j\in\mathbb{N}}$ is increasing as claimed. Since for any $l\in\mathbb{Z}$ it holds $l+1\leq l+s_l$, we have $k+1\in J^i_k$. Therefore α^i_k intervenes in the decomposition (10), so $\alpha^i_{k+1}\geq\alpha^i_k$.

If X is an Enoki surface, then $\alpha_{i+j}=1$ for every $j\geq 0$ as we have seen in Example 4.3. If X is not an Enoki surface there is at least one curve $\widetilde{C}_{i+j}$ such that $\widetilde{C}^2_{i+j}\leq -3$, hence, using again formula (7) we see that $\widetilde{e}_{i+j+2}$ appears in the decompositions of the curves $\widetilde{C}_{i+j}$ and $\widetilde{C}_{i+j+1}$, hence $j,\ j+1\in J^i_{j+2}$ and $\alpha_{j+2}\geq\alpha_j+\alpha_{j+1}>\alpha_{j+1}$. Since the configuration of the curves in $\widetilde{X}$ is periodic, we have $\widetilde{C}^2_{i+j+kn}=\widetilde{C}^2_{i+j}\leq -3$ for every $k\in\mathbb{N}$, so $\alpha_{j+kn+2}>\alpha_{j+kn+1}$, which proves the result. □

Remark 4.7. *The decomposition* $\widetilde{e}_i=\sum_{j\geq 0}\alpha^i_j[\widetilde{C}_{i+j}]$ *given by Proposition 4.6 is the only decomposition of the image of* $\widetilde{e}_i$ *in* $H_2^{\mathrm{BM}}(\widetilde{X},\mathbb{Z})$ *as the sum of a series – bounded towards the pseudo-convex end – of classes of compact curves.*

Proof. Indeed, it suffices to see that if a sum $\sum_{i\geq k}a_i[\widetilde{C}_i]$ vanishes in $H_2^{\mathrm{BM}}(X,\mathbb{Z})$, then all coefficients a_i vanish. This follows easily using the Poincaré duality isomorphism $PD: H_2^{\mathrm{BM}}(\widetilde{X},\mathbb{Z})\xrightarrow{\simeq} H^2(\widetilde{X},\mathbb{Z})$. Using the geometric interpretation of the Poincaré duality in terms of intersection numbers we get

$$0=\langle PD(\sum_{i\geq k}a_i[\widetilde{C}_i]),\widetilde{e}_k\rangle=-a_k \ .$$

By induction we get $a_i=0$ for all $i\geq k$. □

Note that for an Enoki surface X the classes $[\widetilde{C}_i]$ are linearly independent in $H_2(\widetilde{X},\mathbb{Z})$, but $\sum_{i\in\mathbb{Z}}[\widetilde{C}_i]=0$ in $H_2^{\mathrm{BM}}(\widetilde{X},\mathbb{Z})$.

Let now $p:\mathcal{X}\to B$ be a holomorphic family of GSS surfaces and let

$$e=(e_b)_{b\in B}\in H^0(B,\underline{H}_2)\ ,\ \widetilde{e}=(\widetilde{e}_b)_{b\in B}\in H^0(B,\underline{\widetilde{H}}_2)$$

be families of homology classes as in the first extension theorem Theorem 3.1.

Our goal now is to identify explicitly the effective divisor $\widetilde{E}_{b_0}$ for $b_0\in H_e$. As we explain in the introduction this divisor is interesting because it can be written as a limit of simply exceptional divisors $\widetilde{E}_{b_n}$ for a sequence $b_n\to b_0$, $b_n\in B_e$.

We suppose for simplicity that X_{b_0} is minimal because, if not, we can blow down the exceptional divisors in X_{b_0} (and their deformations in X_b for b around b_0) and reduce the problem to the case of a minimal central fiber.

Theorem 4.8. *Under the assumptions and with the notations above, the effective divisor $\widetilde{E}_{b_0}$ is the sum of the infinite series $\sum_{j\geq 0}\alpha_j^i\widetilde{C}_{i+j}$ given by Proposition 4.6.*

Proof. Our extension theorem yields a divisor $\widetilde{\mathcal{E}}\subset\widetilde{\mathcal{X}}$ flat over B whose fiber over b_0 is $\widetilde{E}_{b_0}$, so the class defined by $\widetilde{E}_b$ in Borel-Moore homology is $\widetilde{e}_b$ for any $b\in B$. The result follows directly from Remark 4.7 and Lemma 4.9 below. □

Lemma 4.9. *Let X be a GSS surface with $b_2(X)>0$ and $\alpha:\widetilde{X}\to X$ its universal cover. Then $\widetilde{X}$ does not contain any irreducible non-compact 1-dimensional analytic subset bounded towards the pseudo-convex end.*

Proof. Suppose that there exists such a subspace $S\subset\widetilde{X}$. We can suppose that X is minimal and was identified with $X(\Pi,\sigma)$ as explained above. Since S is bounded towards the pseudo-convex end we can find k sufficiently large such that S does not intersect the closure of $\widetilde{X}_k$.

We have a natural surjective *holomorphic* map $q_k:\widetilde{X}\to\widehat{X}_k$ which contracts all compact curves $\widetilde{C}_l$ for $l>k$ to a point $\widehat{O}_k\in\widehat{X}_k$ (the center of ball $B_{k-n\left[\frac{k}{n}\right]}$ involved in the construction of $\widehat{X}_k$) and defines a biholomorphism

$$\widetilde{X}\setminus\bigcup_{l>k}\widetilde{C}_l\xrightarrow{\simeq}\hat{X}_k\setminus\{\widehat{O}_k\}$$

(see [4]). Since S does not intersect the closure of $\widetilde{X}_k$ the image $S_0:=q_k(S)$ is contained in the complement of this closure in $\hat{X}_k$, which is the open ball $B_{k-n\left[\frac{k}{n}\right]}$.

But then $S_0\setminus\{\widehat{O}_k\}$ is a closed 1-dimensional subspace of a punctured open 2-ball, so its closure in this ball will be a closed 1-dimensional subspace of the ball, by the second Remmert-Stein theorem. This extension would be compact, which is of course impossible.

□

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INFINITE BUBBLING IN NON-KÄHLERIAN GEOMETRY

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Abstract. In a holomorphic family $(X_b)_{b\in B}$ of non-Kählerian compact manifolds, the holomorphic curves representing a fixed 2-homology class do not form a proper family in general. The deep source of this fundamental difficulty in non-Kähler geometry is the *explosion of the area* phenomenon: the area of a curve $C_b \subset X_b$ in a fixed 2-homology class can diverge as $b \to b_0$. This phenomenon occurs frequently in the deformation theory of class VII surfaces. For instance it is well known that any minimal GSS surface X_0 is a degeneration of a 1-parameter family of simply blown up primary Hopf surfaces $(X_z)_{z\in D\setminus\{0\}}$, so one obtains non-proper families of exceptional divisors $E_z \subset X_z$ whose area diverge as $z \to 0$. Our main goal is to study in detail this non-properness phenomenon in the case of class VII surfaces. We will prove that, under certain technical assumptions, a lift $\widetilde{E}_z$ of E_z in the universal cover $\widetilde{X}_z$ does converge to an effective divisor $\widetilde{E}_0$ in $\widetilde{X}_0$, but this limit divisor is not compact. We prove that this limit divisor is always bounded towards the pseudo-convex end of $\widetilde{X}_0$ and that, when X_0 is a minimal surface with global spherical shell, it is given by an infinite series of *compact* rational curves, whose coefficients can be computed explicitly. This phenomenon – degeneration of a family of compact curves to an infinite union of compact curves – should be called *infinite bubbling*.

We believe that such a decomposition result holds for any family of class VII surfaces whose generic fiber is a blown up primary Hopf surface. This statement would have important consequences for the classification of class VII surfaces.

MSC: 32Q65, 32Q57, 32Q55

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1. Introduction: What happens with the exceptional curves in a deformation of class VII surfaces?

1.1. **Properness properties in Kählerian and symplectic geometry.** We will use the following notation: for a complex manifold X and a class $c \in H_{2d}(X,\mathbb{Z})$ we denote by $\mathcal{B}^c(X)$ the Barlet space [1] of d-cycles in X representing the class c.

Let $p : \mathcal{X} \to B$ be a family of compact n-dimensional complex manifolds parameterized by a complex manifold B. In other words, p is a proper holomorphic submersion with connected fibers. We denote by $\underline{H}_m$, $\underline{H}^m$ the locally constant sheaves (coefficient

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systems) on B which are associated with the presheaves $B \supset U \mapsto H_m(p^{-1}(U), \mathbb{Z})$, respectively $B \supset U \mapsto H^m(p^{-1}(U), \mathbb{Z})$. Note that the sheaf $\underline{H}^m$ coincides with the m-th direct image $R^m p_*(\underline{\mathbb{Z}})$ of the constant sheaf $\underline{\mathbb{Z}}$ on $\mathcal{X}$. We fix $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_{2d})$, and we denote by $\mathcal{B}^e_B(\mathcal{X})$ the relative Barlet cycle space associated with e. This space is a closed subspace of the cycle space $\mathcal{B}(\mathcal{X})$ and, as a set, it coincides with the union $\cup_{b\in B}\mathcal{B}^{e_b}(X_b)$. The obvious map

$$\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$$

is holomorphic and its (reduced) fibers are precisely the spaces $\mathcal{B}^{e_b}(X_b)$. A fundamental consequence of Bishop compactness theorem states that in the Kählerian framework the projection $\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$ is also proper. More precisely:

Proposition 1.1. *Let H be a Hermitian metric on $\mathcal{X}$ such that the restrictions to the fibers of the corresponding Kähler form $\Omega_H \in A^{1,1}(\mathcal{X}, \mathbb{R})$ are all closed*[1]*. Then the map $\mathcal{B}^e(p) : \mathcal{B}^e_B(\mathcal{X}) \to B$ is proper.*

Proof. The total space of the locally constant sheaf $\underline{H}^{2d} \otimes \mathbb{R} = R^{2d}p_*(\underline{\mathbb{R}})$ is a real vector bundle on B, and the assignment $b \mapsto [\Omega^d_H\big|_{X_b}]$ defines a smooth section χ of this bundle. For a cycle $C \in B^{e_b}(X_b)$ one has

$$\mathrm{vol}_H(C) = \int_C \Omega^d_H = \langle \chi(b), e_b\rangle \ .$$

Therefore the volume map $\mathrm{vol}_H : \mathcal{B}^e_B(\mathcal{X}) \to \mathbb{R}_{\geq 0}$ can be written as $\mathcal{B}^e(p)^*\langle \chi, e\rangle$, so it is bounded on $\mathcal{B}^e(p)^{-1}(K)$ for every compact set $K \subset D$. It suffices to apply the relative-compactness criterion given by Theorem 1 of [1]. □

This properness result can be intuitively interpreted as follows:

The cycles in a fixed homology class of a compact complex manifold X_b cannot disappear as X_b tends to a limit compact Kähler manifold X_{b_0}.

The properness property has a particular signification in the case $n = 2$, $d = 1$. Indeed, in this case the Barlet space $\mathcal{B}^e(X_b)$ is just a moduli space of effective divisors representing the class e, and this moduli space can be identified with the moduli space of Seiberg-Witten monopoles associated with the e-twisted canonical Spin^c-structure of X_b and a suitable perturbation of the monopole equation (see [17], [11], [16]). Therefore, in this case, the properness property above is a consequence of the general properness property – with respect to the space of parameters – for moduli spaces of Seiberg-Witten monopoles, which is crucial for proving that Seiberg-Witten invariants are well-defined.

A similar properness result – based on a version of Gromov compactness theorem – holds in symplectic geometry: let $\mathcal{X} \to B$ be a proper smooth map with connected fibers in the differentiable category, suppose that Ω is a smooth 2-form on $\mathcal{X}$ whose restrictions to the fibers $\omega_b := \Omega\big|_{X_b}$ are all symplectic forms (in particular closed), and consider a smooth family $(J_b)_{b\in B}$ of almost complex structures on the fibers, such that J_b is compatible with ω_b for any $b \in B$, and a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$. Let $(b_n)_{n\in\mathbb{N}^*}$ be a sequence in $B \setminus \{b_0\}$ converging to b_0 and consider, for every $n \in \mathbb{N}$, an almost holomorphic map

$$\varphi_n : (\Sigma, j_n) \to (X_{b_n}, J_{b_n})$$

[1]We can always construct such a Hermitian metric in a neighborhood of a fixed fiber X_{b_0} if this fiber is Kählerian. It suffices to consider a smooth family of Kähler metrics h_b on X_b (using the openness property of the Kähler condition) for b close to b_0, and to add the pull-back of a Hermitian metric on the base.

representing $e_{b_n} \in H_2(X_{b_n}, \mathbb{Z})$, where j_n is an almost holomorphic structure on a differentiable closed surface Σ. Since the map $b \mapsto [\omega_b] \in H^2(X_b, \mathbb{R})$ defines a smooth section in the vector bundle $\underline{H}^2 \otimes \mathbb{R}$, we get a bound of the area sequence

$$\left(\int_\Sigma \varphi_n^*(\omega_{b_n}) \right)_{n\in\mathbb{N}} = \left(\langle [\omega_{b_n}], e_{b_n} \rangle \right)_{n\in\mathbb{N}} .$$

Therefore (using the terminology and the main result of [12]) there exists a subsequence (b_{n_k}) of $(b_n)_{n\in\mathbb{N}}$ such that that the sequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ converges in the $\mathcal{C}^m$-topology to a *cusp pseudo-holomorphic curve*

$$\varphi : \cup_l \Sigma_l \to X_{b_0} .$$

A similar convergence result can be proved for sequences of cusp curves. In other words, in symplectic geometry – as in Kählerian geometry – pseudo-holomorphic curves representing a fixed homology class can degenerate, but *they cannot disappear in a limit process.* An important particular case is the one when $j_n = j$ is independent of n. In this case it is known that the domain $\cup_l \Sigma_l$ of the limit cusp curve is obtained from Σ by adding a union of *bubble trees* of 2-spheres. In particular, when $(\Sigma, j) = \mathbb{P}^1$, all the irreducible components of the limit cusp curve will be 2-spheres.

These properness results and their natural generalizations for perturbed pseudo-holomorphic curves, play a crucial role in proving that (different versions of) Gromov-Witten invariants are well-defined.

For deformations of compact non-Kählerian manifolds the analogue properness results do not hold. *The properness property fails even in the cases when all cycle spaces $\mathcal{B}^e(X_b)$ are compact.* This is one of the main difficulties in non-Kählerian geometry, which has dramatic consequences, for instance: in non-Kählerian geometry one cannot use enumerative methods of curves to construct deformation invariants, and this *even if one considers only moduli spaces of curves which are finite for all manifolds in the considered deformation class!* In non-Kählerian geometry holomorphic curves representing a given homology class *can* disappear in a limit process, because the area of the curves in a given homology class is not a priori bounded, so it can explode to infinity when one approaches a fixed fiber X_{b_0}. Probably the most convincing examples which illustrates this difficulty (and its dramatic consequences) appear in the deformation theory of class VII surfaces.

1.2. **Deformations of class VII surfaces. What happens with the exceptional curves?** Let X be a class VII surface with $b_2 := b_2(X) > 0$ admitting a global spherical shell (a GSS), see section 4 for details. We will call such a surface a GSS surface. For such a surface one has $\pi_1(X, x_0) \simeq \mathbb{Z}$, and the universal cover $\widetilde{X}$ has two ends, a pseudo-convex end and a pseudo-concave end. Taking a non-separating strictly pseudo-convex embedded 3-sphere $\Sigma \subset X$, consider a lift A of $X \setminus \Sigma$ in $\widetilde{X}$ and note that $\bar{A}$ is a compact surface bounded by two lifts of Σ to $\widetilde{X}$; the pseudo-convex end of $\widetilde{X}$ corresponds to the pseudo-convex boundary component of A [2].

For such a surfaces one also has $H_2(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{b_2}$, and the intersection form on this group is standard, i.e. there exists a basis $(e_0, \ldots, e_{b_2-1})$ in $H_2(X, \mathbb{Z})$ such that $e_i \cdot e_j = -\delta_{ij}$. Changing signs if necessary one can assume that

$$-c_1(X) = c_1(\mathcal{K}_X) = \sum_{i=0}^{b_2-1} e_i ,$$

and the basis obtained in this way is unique up to order.

[2]Note that the terminology used here for the two ends of $\widetilde{X}$ is different from the one used in [4], but agrees with the terminology used in the theory of complete, non-compact complex manifolds.

By results of Kato and Dloussky it is well-known that any minimal GSS surface X_0 is deformable in blown up primary Hopf surfaces. More precisely, there exists a holomorphic family $\mathcal{X} \to D$ parametrized by the unit disk $D \subset \mathbb{C}$ having X_0 as central fiber such that $X_z \simeq [\hat{H}_z]_{P(z)}$ is biholomorphic to a primary Hopf surface $H(z)$ blown up in a finite set of simple points $P(z) = \{p_0(z), \dots, p_{b_2-1}(z)\}$. Therefore, for any $z \neq 0$, the fiber X_z has b_2 exceptional curves $E_{0,z}, \dots, E_{b_2-1,z}$ representing the classes e_i. On the other hand the central fiber X_0 is minimal hence, by a result of Nakamura (see [10], Lemma 1.1.3]), it does not admit any effective divisor at all representing a class e_i. This shows that the rational curve $E_{i,z}$ has no limit at all as $z \to 0$, even if one admits cusp curves (and bubbling trees) in the limit process.

Note that, for $z \neq 0$, the exceptional curve $E_{i,z}$ is a regular point in the moduli space of divisors representing the class e_i on X_z, so if one tries to define a Gromov-Witten type "invariant" of X_z by counting the rational curves in this homology class, the result will be 1. But the same "invariant" will be 0 for X_0.

Similar phenomena appear for deformations of minimal GSS surfaces. For instance, there exists a family $\mathcal{X} \to D$ of minimal GSS surfaces with $b_2 = 1$ whose generic fiber X_z ($z \neq 0$) has only one irreducible curve C_z, which is homologically trivial, whereas X_0 has only one irreducible curve D, which represents the class $-e_0$. In this family all curves are singular rational curves with a simple node. In both cases one can study explicitly the evolution of the area of the curves $E_{i,z}$, C_z and see that their area tends to infinity as $z \to 0$.

The conclusion is clear: one cannot hope to define deformation invariants for class VII surfaces by counting holomorphic curves in a given homology class. This is one of the major difficulties in the classification of class VII surfaces, because the structure of curves on these surfaces is related to the fundamental classification problem. Indeed, by the main result of [6] one knows that the global spherical conjecture (which, if true, would complete the classification of class VII surfaces) reduces to the conjecture:

C: *Any minimal class VII surface X with $b_2(X) > 0$ has $b_2(X)$ rational curves.*

Therefore, proving existence of curves is a fundamental problem in the theory of class VII surfaces. Unfortunately the examples above show that one cannot hope to prove such existence results using invariants of Gromov-Witten type. Note however that, taking into account this difficulty (the lack of Gromov-Witten type invariants), the conjecture above becomes quite intriguing. Indeed, this conjecture (which is true for all GSS surfaces) implies that the total number of rational curves *is* an invariant for class VII surfaces, although this invariant cannot be obtained as the sum of Gromov-Witten type invariants defined for individual homology classes (because, as we have just seen, the number of rational curves in an individual class is *not* an invariant). In other words, although the homology classes represented by rational curves change in a holomorphic family, the total number of rational curves remains always constant in deformations of known class VII surfaces. In particular, when (in a limit process) a homology class loses a rational curve, there should be always another class which gets one. This "global compensation phenomenon" between different moduli spaces of rational curves is not understood yet, so conjecture **C** is not known yet even for surfaces which are degenerations of the known GSS surfaces.

Although the cause of the "curve vanishing " (non-properness) phenomenon is clear – explosion of the area in the limit process – one can still wonder when contemplating the first family described above:

"What happens with the exceptional curves $E_{i,z}$ when $z \to 0$?"

The authors have many times been asked this question when they gave talks on class VII surfaces in the past. The easy answer we have always given

"They just disappear, because their area explodes",

although perfectly correct, has never seemed to fully satisfy the audience. Therefore we decided to investigate in detail this phenomenon trying to understand what happens geometrically with the exceptional curve $E_{i,z}$ as it gets bigger and bigger.

We came to the idea of an "infinite bubbling tree" empirically, when the second author noticed that recent results of the first author [5] have an intriguing consequence: Although for a minimal GSS surface X_0 the homology class e_i is not represented by an effective divisor (as we saw above), any of its lifts $\widetilde{e}_i$ in the universal cover $\widetilde{X}_0$ is represented – in the homology with closed supports – by an infinite series of compact curves. Moreover, in every case we obtained a well defined series representing the given lift, and we saw that always this series escapes to infinity towards the pseudo-concave end of $\widetilde{X}_0$. Therefore we came to idea to prove that (although $E_{i,z}$ accumulates to the whole X_0 as $z \to 0$ by the Remmert-Stein theorem) a fixed lift $\widetilde{E}_{i,z}$ of $E_{i,z}$ in the universal covering $\widetilde{X}_z$ of X_z behaves in a controllable way as $z \to 0$, namely that

Theorem 1.2. *The family of rational curves $(\widetilde{E}_{i,z})_{z\in D\setminus\{0\}}$ fits in a flat family $\widetilde{\mathcal{E}}_i$ of effective divisors over the whole base D. In particular this family has a limit as $z \to 0$, which is given by an infinite series of compact curves which escapes to infinity towards the pseudo-concave end of $\widetilde{X}_0$.*

The main purpose of this article is to prove this result. More precisely we will prove that the limit given by the theorem is precisely the series representing $\widetilde{e}_i$ we discovered empirically. We called this phenomenon "infinite bubbling", and we believe that this happens in all deformations of class VII surfaces when area explosion occurs. In fact we will prove a more general result:

Theorem 1.3. *Let $\mathcal{X} \to \Delta$ be any holomorphic family of GSS surfaces with $b_2 > 0$, parameterized by the standard ball $\Delta \subset \mathbb{C}^r$. Then for any lift $\widetilde{e}_i$ of a homology class e_i to the universal cover $\widetilde{\mathcal{X}}$, there exists an effective divisor $\widetilde{\mathcal{E}}_i \subset \widetilde{\mathcal{X}}$ flat over Δ such that*

1. *The fiber $\widetilde{E}_{i,z} \subset \widetilde{X}_z$ of $\widetilde{\mathcal{E}}_i$ over a point $z \in \Delta$ is a lift of the exceptional curve E_i representing e_i, for every $z \in \Delta$ for which X_z contains such an exceptional curve.*
2. *If X_z does not admit any exceptional effective divisor in the class e_i then the corresponding fiber $\widetilde{E}_{i,z}$ of $\widetilde{\mathcal{E}}_i$ is a series of compact curves which escapes to infinity towards the pseudo-concave end of $\widetilde{X}_z$. This divisor represents the image of the class $\widetilde{e}_i$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(X, \mathbb{Z})$.*

Finally, we can generalize this result for deformations of an *unknown* minimal class VII surface which can be deformed in GSS surfaces:

Theorem 1.4. *Let $\mathcal{X} \to \Delta$ be a holomorphic family such that X_z is a GSS surface with $b_2 > 0$ for any $z \in \Delta \setminus A$, where $A \subset \Delta$ is an analytic subset of codimension ≥ 2, and X_a is minimal for every $a \in A$. Then for any lift $\widetilde{e}_i$ of a homology class e_i to the universal cover $\widetilde{\mathcal{X}}$, there exists an effective divisor $\widetilde{\mathcal{E}}_i \subset \widetilde{\mathcal{X}}$ flat over Δ such that the fiber $\widetilde{E}_{i,z} \subset \widetilde{X}_z$ of $\widetilde{\mathcal{E}}_i$ over a point $z \in \Delta$ has the properties:*

1. *it represents the image of the class $\widetilde{e}_i$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(X, \mathbb{Z})$ for every $z \in \Delta$,*
2. *is a lift of the exceptional curve E_i representing e_i, for every $z \in \Delta$ for which X_z contains such an exceptional curve.*

Unfortunately we did not succeed to prove that the second statement of Theorem 1.3 also holds in this case, so we cannot prove that, for $a \in A$, the limit divisor $\widetilde{E}_{i,a}$ is given by a series of *compact* curves. The difficulty is to rule out the appearance of an

open Riemann surface in the limit. Such a result would be of crucial importance for the classification of class VII surfaces, because it will prove the main conjecture **C** for any class VII surface which fits as the central fiber of a bidimensional family $\mathcal{X} \to \Delta \subset \mathbb{C}^2$ whose fibers X_z, $z \neq 0$ are GSS surfaces (see Theorem 4.10).

2. The non-minimality divisors

A compact effective divisor D on a (non-necessary compact) complex surface X will be called *simply exceptional* if there exists a *smooth* surface Y, a point $y \in Y$, and a surjective holomorphic map $c : X \to Y$ such that $c_{\lfloor X \setminus D} : X \setminus D \to Y \setminus \{y\}$ is biholomorphic, and the fiber $c^{-1}(y)$ over y is D (regarded as a complex subspace of X). If this is the case, the pair (X, D) can be obtained from Y by iterated blowing up at y. More precisely (X, D) is the last term (X_n, D_n) in a sequence $(X_1, D_1), \ldots, (X_n, D_n)$ $(n \geq 1)$, where

- X_1 is the blow up of Y at y and D_1 is the corresponding exceptional divisor,
- X_k is obtained from X_{k-1} by blowing up at a point $y_{k-1} \in D_{k-1}$ for $2 \leq k \leq n$, and D_k is the pre-image of D_{k-1} in X_k.

Therefore D is the pre-image in X_n of the exceptional curve of first kind $D_1 \subset X_1$, and it is a tree of smooth rational curves containing at least one exceptional curve of the first kind.

Remark 2.1. *For simply exceptional divisor $D \subset X$ one has*

$$(D, \mathcal{K}_X) = (D, \mathcal{N}_{D/X}) = -1 \ .$$

Using the duality isomorphism $PD : H_2(X, \mathbb{Z}) \simeq H^2_c(X, \mathbb{Z})$ we have

$$D^2 := \langle PD([D]), [D] \rangle = -1 \ .$$

Since the arithmetic genus $g_a(D)$ vanishes, one obtains $h^0(\mathcal{O}_D(D)) - h^1(\mathcal{O}_D(D)) = 0$ by Riemann-Roch theorem for embedded curves ([2], p. 65). Therefore, since $\deg_D(\mathcal{O}_D(D)) = -1 < 0$, we get

$$h^0(\mathcal{O}_D(D)) = h^1(\mathcal{O}_D(D)) = 0 \ ,$$

so, denoting by $e \in H_2(X, \mathbb{Z})$ the homology class represented by D, it follows that the Douady space $\mathcal{D}ou^e(X)$ of compact effective divisors representing this class is 0-dimensional at D, and D is a smooth point in this space (see [3] p. 135).

Remark 2.2. *Let X be a compact surface and D, E simple exceptional divisors representing the same rational homology class $e \in H_2(X, \mathbb{Q})$. Then $D = E$.*

Proof. (X, D) is the last term (X_n, D_n) of a sequence of pairs $(X_1, D_1), \ldots, (X_n, D_n)$ obtained from (Y, y) as above. We prove the statement by induction on n. If $n = 1$, D is an exceptional curve of the first kind. Since $DE = -1$, it follows that D is one of the irreducible components of E. But the $\mathbb{Q}$-homology classes of the irreducible components of a simply exceptional divisor are linearly independent, so one can have $[D]_{\mathbb{Q}} = [E]_{\mathbb{Q}}$ if and only if $D = E$.

If $n > 1$, let D' be the exceptional curve of the first kind of the blowing up $X_n \to X_{n-1}$. One has obviously $D'D = 0$, so $D'E = 0$. Therefore either D' is an irreducible component of E, or $D' \cap E = \emptyset$. In both cases the direct images $(p_n)_*(D)$, $(p_n)_*(E)$ via the contraction map $p_n : X_n \to X_{n-1}$ are simply exceptional divisor in the rational homology class $(p_n)_*(e)$, so they coincide by induction assumption. On the other hand one can write $D = p_n^*((p_n)_*(D))$, $E = p_n^*((p_n)_*(E))$, so $D = E$. □

The condition "X admits a simply exceptional effective divisor D" is open with respect to deformations of X. More precisely:

Proposition 2.3. *Let $p : \mathcal{X} \to B$ be a holomorphic submersion with $\dim(\mathcal{X}) = \dim(B) + 2$. Let $D_0 \subset X_{b_0}$ be a simply exceptional divisor, and $e \in H_2(\mathcal{X}, \mathbb{Z})$ its homology class in $\mathcal{X}$. There exists a neighborhood $\mathcal{U}$ of D_0 in $\mathcal{D}ou^e(\mathcal{X})$ such that*

1. *for every $D \in \mathcal{U}$, $p(D)$ is a singleton denoted $\{\mathfrak{p}(D)\}$ in B and D is a tree of rational curves in $X_{\mathfrak{p}(D)}$,*
2. *the induced map $\mathfrak{p} : \mathcal{U} \to B$ is a local biholomorphism at D_0,*
3. *D is simply exceptional in $X_{\mathfrak{p}(D)}$ for every $D \in \mathcal{U}$*

Proof. Let $V \subset B$ an open neighborhood of b_0 which is the domain of a chart, and $\mathcal{V} \subset \mathcal{D}ou^e(\mathcal{X})$ the neighborhood of D_0 in $\mathcal{D}ou^e(\mathcal{X})$ consisting of 1-dimensional complex subspaces of $\mathcal{X}$ which represent the class e and are contained in $p^{-1}(V)$.

For every $D \in \mathcal{V}$ the projection $p(D) \subset V$ is a finite union of points. On the other hand it's easy to see that any small flat deformation of a tree of smooth rational curves is again a tree of smooth rational curves, in particular it is connected. Therefore, replacing $\mathcal{V}$ by a smaller neighborhood if necessary, the conclusion (1) will be fulfilled.

The normal bundle $\mathcal{N}_{\mathcal{X}}(D_0)$ of D_0 in $\mathcal{X}$ fits in a short exact sequence on D_0:

$$0 \to \mathcal{N}_{X_{b_0}}(D_0) = \mathcal{O}_{D_0}(D_0) \to \mathcal{N}_{\mathcal{X}}(D_0) \to D_0 \times T_{b_0}(B) \to 0 \ .$$

Since $h^1(\mathcal{O}_{D_0}(D_0)) = h^1(\mathcal{O}_{D_0}) = 0$ we obtain

$$H^0(\mathcal{N}_{\mathcal{X}}(D_0)) = T_{b_0}(B) \ , \ H^1(\mathcal{N}_{\mathcal{X}}(D_0)) = \{0\} \ .$$

This shows that $\mathcal{D}ou^e(\mathcal{X})$ is smooth at D_0 and the projection on B defined by $\mathfrak{p}$ is a local biholomorphism at D_0. This proves (2). Let $\mathcal{U} \subset \mathcal{V}$ be an open neighborhood of D_0 on which $\mathfrak{p}$ is biholomorphic, and put $U := \mathfrak{p}(\mathcal{U}) \subset B$. We obtain a holomorphic family of trees of rational curves $(D_b)_{b \in U}$ with $D_{b_0} = D_0$.

The third statement can be proved by induction with respect to the number m of irreducible components of D_0. Consider an irreducible component D_m of D_0 which is an exceptional curve of the first kind. D_m is itself a simply exceptional divisor, so the statements (1), (2) apply to D_m, and, taking $\mathcal{U}$ and U sufficiently small, we obtain a holomorphic family

$$(D_{m,b})_{b \in U} \ , \ D_{m,b} \subset X_b$$

of simply exceptional divisors. But, by the stability theorem of Kodaira, any small deformation of an exceptional curve of the first kind is again an exceptional curve of the first kind, so we may assume that all $D_{m,b}$ are all exceptional curves of the first kind. This proves the statement when $m = 1$. Suppose $m > 1$. Contracting in X_b the curve $D_{m,b}$ to a point y_b for every $b \in B$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_U & \xrightarrow{\pi} & \mathcal{Y} \\ & {\scriptstyle p_U}\searrow \quad \swarrow{\scriptstyle s} & \\ & U & \end{array}$$

where π is proper and bimeromorphic whose restriction to any fiber X_b is just the blowing down map $\pi_b : X_b \to Y_b$ which contracts $D_{m,b} \subset X_b$ to y_b. Let $\mathcal{D}$, $\mathcal{D}_m \subset \mathcal{X}_U$ be the classifying divisors of the families $(D_b)_{b \in U}$, $(D_{m,b})_{b \in U}$. Supposing U connected, we see that $\mathcal{D}_m$ is irreducible. One has $(\mathcal{D} \cap \mathcal{D}_m) \cap X_{b_0} = D_{m,b_0}$, so (since we supposed $m > 1$) one has $\mathcal{D} \neq \mathcal{D}_m$. Let $\mathcal{D}'$ be the divisor which coincides with $\mathcal{D}$ if $\mathcal{D}_m \not\subset \mathcal{D}$ and is obtained from $\mathcal{D}$ by removing the $\mathcal{D}_m$ if $\mathcal{D}_m \subset \mathcal{D}$. The projection on $\mathcal{Y}$ of every irreducible component of $\mathcal{D}'$ is reduced and irreducible of codimension 1, so $\mathcal{D}'' = \pi(\mathcal{D}')$ is an effective divisor of $\mathcal{Y}$. Any fiber D''_b of this divisor over a point $b \in U$ is an effective divisor of Y_b (more precisely a tree of rational curves in Y_b) so $\mathcal{D}''$ is flat over U by Lemma 2.4 below. By induction assumption all fibers D''_b are simply exceptional divisors. Note now that D_b is the pre-image $\pi_b^*(D''_b)$ of D''_b in X_b. Therefore D_b will be simply exceptional, too. □

Lemma 2.4. *Let $p : \mathcal{X} \to B$ be a submersion of complex manifolds with $\dim(\mathcal{X}) = n$, $\dim(B) = m$. Let $\mathcal{D} \subset \mathcal{X}$ be an effective divisor such that $\mathcal{D} \cap X_b$ is a (possibly empty) effective divisor of the fiber X_b for every $b \in B$. Then $\mathcal{D}$ is flat over B.*

Proof. Let $x \in \mathcal{D}$ and let $\phi : U \to \mathbb{C}$ be a local equation for $\mathcal{D}$ around x, i.e. U is an open neighborhood of x in $\mathcal{X}$ and $\mathcal{D} \cap U = Z(\phi)$. Consider the map $\psi := (p, \phi) : U \to B \times \mathbb{C}$, and note that the fiber $\psi^{-1}(\psi(x))$ of this map can be identified with $\mathcal{D} \cap U \cap X_{p(x)}$ so the dimension at x of this fiber has minimal value $n - m - 1$. By the fiber dimension semicontinuity theorem we see that, replacing U with a smaller open neighborhood if necessary, we may assume that $\dim_\xi \psi^{-1}(\psi(\xi)) = n - m - 1$ for every $\xi \in U$. Therefore ψ will be flat by the flatness criterion given by the Corollary on p. 158 in [8]. Now we apply the base change property of the flat morphisms to the base change $B \ni b \mapsto (b, 0) \in B \times \mathbb{C}$ and we get that $\mathcal{D} \cap U$ is flat over B. □

For a holomorphic family $\mathcal{X} \to B$ of compact complex surfaces we will denote by $k = (k_b)_{b \in B} \in H^0(B, \underline{H}^2)$ the section defined by the classes $k_b := c_1(\mathcal{K}_{X_b})$, $b \in B$.

Corollary 2.5. *Let $\mathcal{X} \to B$ be a holomorphic family of compact complex surfaces, and let $e = (e_b)_{b \in B} \in H^0(B, \underline{H}_2)$ such that $\langle e, k \rangle = -1$ and $e \cdot e = -1$ with respect to the intersection form induced from any $H_2(X_b, \mathbb{Z})$. Then the subset*

$$B_e := \{b \in B |\ X_b \text{ admits a simply exceptional divisor representing the class } e_b\} \subset B$$

is open with respect to the classical topology.

Lemma 2.6. *Let S be a class VII surface and let C be an effective divisor in X with*

$$C^2 = (C, \mathcal{K}_S) = -1 \ .$$

Then $C = E + \Gamma$, where E is a simply exceptional divisor, $p \geq 0$, and $\Gamma = f^\star \Gamma'$ is the pre-image of a homologically trivial effective divisor $\Gamma' \subset S'$ via the contraction $f : S \to S'$ to the minimal model.

Proof. If the minimal model of S is an Inoue surface S' with $b_2(S') = 0$, the result is clear, because an Inoue surface with $b_2 = 0$ does not contain any curve. Therefore we shall suppose that the minimal model is a Hopf surface or a minimal class VII surface with $b_2 > 0$. Let $C = \sum_{i=1}^m n_i C_i$ $(n_i > 0)$ be the decomposition of C as linear combination of irreducible components. We prove the result by induction on the sum $\sigma_C := \sum_{i=1}^m n_i \geq 1$. If $\sigma_C = 1$, then $C = C_1$ is irreducible with $C_1^2 = (C, \mathcal{K}_S) = -1$, so it is an exceptional curve of the first kind by Proposition 2.2 in [2].

Suppose now $\sigma_C \geq 2$. We have

$$-1 = (C, \mathcal{K}_S) = \sum_i n_i (C_i, \mathcal{K}_S)$$

hence there exists an index j such that $(C_j, \mathcal{K}_S) < 0$. Since the intersection form of a class VII surface is negative definite, one has $C_j^2 \leq 0$, with equality if and only if C_j is homologically trivial. This would contradict $(C_j, \mathcal{K}_S) < 0$, so necessary $C_j^2 < 0$. By Proposition 2.2 in [2] again it follows that C_j is an exceptional curve of the first kind.

Case 1. The homology classes $[C]_{\mathbb{Q}}$, $[C_j]_{\mathbb{Q}}$ coincide in $H_2(S, \mathbb{Q})$. In this case the effective divisor $\Gamma := C - C_j$ is $\mathbb{Q}$-homologically trivial. Therefore the effective divisor $\Gamma' := f_*(\Gamma) \subset S'$ is $\mathbb{Q}$-homologically trivial, too. We can write $\Gamma = f^*(\Gamma') + F$, where F is an effective divisor contained in the exceptional divisor of the contraction map f. But F must be also $\mathbb{Q}$-homologically trivial, so it is empty. This shows that $\Gamma = f^*(\Gamma')$, where Γ' is a $\mathbb{Q}$-homologically trivial effective divisor in the minimal model S'. But, using Enoki's theorem concerning the classification of class VII surfaces admitting a numerically trivial divisor, it follows that on such a surface any $\mathbb{Q}$-homologically trivial effective divisor is homologically trivial.

Case 2. The homology classes $[C]_{\mathbb{Q}}$, $[C_j]_{\mathbb{Q}}$ are different. We contract the exceptional

curve C_j obtaining a blowing down map $g : S \to S_0$. The homology group $H_2(S, \mathbb{Z})$ has a decomposition

$$H_2(S, \mathbb{Z}) = \mathbb{Z}[C_j] \oplus H^\perp \ ,$$

which is orthogonal with respect to the intersection form q_S of S, and g_* induces an isomorphism

$$(H^\perp, q_S|_{H^\perp \times H^\perp}) \xrightarrow{\simeq g_*|_{H^\perp}} (H_2(S_0, \mathbb{Z}), q_{S_0}) \tag{1}$$

Since the rational intersection form $q_S^{\mathbb{R}}$ is negative definite, we have

$$0 \leq |CC_j| = \left|q_S^{\mathbb{R}}([C]_{\mathbb{R}}, [C_j]_{\mathbb{R}})\right| \leq \frac{1}{2}(-C^2 - C_j^2) = 1 \ , \tag{2}$$

with equality if and only if $[C]_{\mathbb{R}}, [C_j]_{\mathbb{R}}$ are colinear. But since these classes belong to the lattice $\mathrm{im}(H_2(S, \mathbb{Z}) \to H_2(S, \mathbb{R}))$ and $(C_j, \mathcal{K}_S) = (C, \mathcal{K}_S) = -1$, the colinearity condition implies $[C]_{\mathbb{R}} = [C_j]_{\mathbb{R}}$, which cannot hold in our case. Therefore the inequality (2) implies $CC_j = 0$, i.e. $[C] \in H^\perp$. Using the isomorphism (1) we see that $g_*(C)^2 = C^2 = -1$. On the other hand $\mathcal{K}_S = g^*(\mathcal{K}_{S_0})(C_j)$, so

$$-1 = (C, \mathcal{K}_S) = (C, g^*(\mathcal{K}_{S_0})(C_j)) = CC_j + (C, g^*(\mathcal{K}_{S_0})) = (g_*(C), \mathcal{K}_{S_0}) \ .$$

This shows that the effective divisor $C_0 := g_*(C)$ of S_0 satisfies the conditions

$$C_0^2 = (C_0, \mathcal{K}_{S_0}) = -1 \ .$$

Since $\sigma_{C_0} < \sigma_C$ we can apply the induction assumption to C_0. Note that $C = g^*(C_0)$. To see this it suffices to see that

- they differ by a multiple of C_j,
- the corresponding homology classes coincide, because they belong to $H^\perp$ and have the same image via g_*.

By induction assumption we know that C_0 decomposes as $C_0 = E_0 + f_0^*(\Gamma_0)$, where $f_0 : S_0 \to S'$ is the contraction to the minimal model of S_0 (which is also the minimal model of S), and Γ_0 is a homologically trivial divisor in S_0. Using the equality $C = g^*(C_0)$, we get $C = g^*(E_0) + g^*(f_0^*(\Gamma_0))$. It suffices to note that $g^*(E_0)$ is a simply exceptional divisor. □

The minimal class VII surfaces admitting a (non-empty) homologically trivial effective divisor Γ are classified [7], we can describe explicitly the second term $f^*(\Gamma)$ in the decomposition given by Lemma 2.6.

Remark 2.7. *If Γ is a (non-empty) homologically trivial effective divisor on a minimal class VII surface X, then the pair (X, Γ) is one of the following:*

1. *X is an elliptic Hopf surface and $\Gamma = \sum_{i=1}^k n_i C_i$, where $n_i \geq 0$, $\sum_{i=1}^k n_i > 0$ and C_i are elliptic curves,*
2. *X is a Hopf surface with two elliptic curves C_1, C_2, and $\Gamma = n_1 C_1 + n_2 C_2$ with $n_i \geq 0$, $n_1 + n_2 > 0$,*
3. *X is a Hopf surface with an elliptic curve C and $\Gamma = nC$ with $n > 0$,*
4. *X is an Enoki surface and $\Gamma = nC$, where $n > 0$ and $C \subset X$ is a cycle of rational curves.*

Lemma 2.8. *Let $\mathcal{X} \to B$ be a holomorphic family of class VII surfaces, let $e = (e_b)_{b \in B} \in H^0(B, \underline{H}_2)$, $b_0 \in B$ and $(b_n)_{n \in \mathbb{N}^*}$ a sequence in B converging to b_0. Consider, for every $n \in \mathbb{N}^*$, a simply exceptional divisor $D_n \subset X_{b_n}$ in the class $e_{b_n} \in H_2(X_{b_n}, \mathbb{Z})$ such that the sequence $(D_n)_{n \in \mathbb{N}^*}$ converges in $\mathcal{B}_B^e(\mathcal{X})$ to a 1-cycle $D_0 \subset X_{b_0}$. Then*

1. *The limit cycle D_0 is a simply exceptional divisor in X_{b_0}.*
2. *Regarding D_n, D_0 as complex subspaces of $\mathcal{X}$, one has $\lim_{n \to \infty} D_n = D_0$ in the Douady space $\mathcal{D}ou(\mathcal{X})$.*

Proof. By Lemma 2.6 we see that D_0 decomposes as $D_0 = E_0 + f_0^*(\Gamma_0)$, where E_0 is a simply exceptional divisor in the class e, $f_0 : X_{b_0} \to X'_{b_0}$ is the contraction to the minimal model, and Γ_0 is a homologically trivial effective divisor in X'_{b_0}. By Proposition 2.3 there exists an open neighborhood $\mathcal{U}$ of E_0 in $\mathcal{D}ou(\mathcal{X})$ such that every element $E \in \mathcal{E}$ is a simply exceptional divisor in the fiber $X_{\mathfrak{p}(D)}$ where $\mathfrak{p} : \mathcal{U} \to U$ is a biholomorphism on an open neighborhood U of b_0. For sufficiently large n one has $b_n \in U$ we obtain a simply exceptional divisor $E_n := \mathfrak{p}^{-1}(b_n)$ in X_{b_n}, and obviously

$$\lim_{n\to\infty} E_n = E_0$$

in the Douady space $\mathcal{D}ou(\mathcal{X})$, so also in the Barlet space $\mathcal{B}^e_B(\mathcal{X})$. But, for every $n \in \mathbb{N}^*$, D_n and E_n are simply exceptional divisors representing the same homology class, so by Remark 2.2 we have $D_n = E_n$ for any $n \in \mathbb{N}^*$. But the Barlet cycle-space is Hausdorff so, since the limit of a convergent sequence in a Hausdorff space is unique, $D_0 = E_0$. This shows that D_0 is a simply exceptional divisor, and the that $\lim_{n\to\infty} D_n = D_0$ also holds in the Douady space. □

Let $p : \mathcal{X} \to B$ be a deformation of class VII surfaces parameterized by a complex manifold B. Let again $k \in H^0(B, \underline{H}^2)$ be the element defined by the family of Chern classes $k_b := c_1(\mathcal{K}_{X_b})$, $b \in B$. Using the exponential short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on $\mathcal{X}$, the vanishing of the cohomology groups $H^2(X_b, \mathcal{O}_{X_b})$ we obtain the exact sequence of sheaves on B

$$0 \to {R^1p_*(\mathcal{O}_\mathcal{X})}/{\underline{H}^1} \to R^1p_*(\mathcal{O}^*_\mathcal{X}) \to \underline{H}^2 \to 0 \ . \tag{3}$$

The quotient sheaf on the left (respectively the sheaf in the middle) is the sheaf of holomorphic sections in a locally trivial complex Lie group fiber bundle Pic^0 (respectively Pic) over B whose fiber over $b \in B$ is $\mathrm{Pic}^0(X_b)$ (respectively $\mathrm{Pic}(X_b)$).

On a class VII surface the canonical morphism $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$ is an isomorphism, so $R^1p_*(\mathcal{O}_\mathcal{X})$ can be identified with $\underline{H}^1 \otimes \mathcal{O}_B$. It is easy to see that the sheaf $\underline{H}^1$ (which is obviously locally constant with fiber isomorphic to $\mathbb{Z}$) is in fact constant; it suffices to see that the degree maps associated with a smooth family of Gauduchon metrics on the fibers define canonical orientations of the real lines $H^1(X_b, \mathbb{R}_{>0}) \subset H^1(X_b, \mathbb{C}^*)$, so canonical orientations of the lines $H^1(X_b, \mathbb{R})$, so canonical generators of the infinite cyclic groups $H^1(X_b, \mathbb{Z})$. Therefore the complex Lie group bundle Pic^0 is in fact trivial with fiber $\mathbb{C}^*$. It is convenient to choose a trivialization $\phi = (\phi_b : \mathbb{C}^* \to \mathrm{Pic}^0(X_b))_{b\in B}$ of this fiber bundle such that for every $b \in B$ one has

$$\lim_{z\to 0} \deg_g(\phi_b(z)) = \infty$$

with respect to a Gauduchon metric on a fiber X_b.

For every section $c \in H^0(B, \underline{H}^2)$ we obtain a subbundle Pic^c of Pic, which becomes a principal $\mathbb{C}^*$-bundle via the chosen trivialization $\phi : B \times \mathbb{C}^* \to \mathrm{Pic}^0$. The associated line bundle $\mathcal{P}^c$ on B can be obtained by adding formally to Pic^c a zero section $\{0_b |\ b \in B\}$, such that for every $b \in B$ one has

$$\forall \lambda \in \mathrm{Pic}^{c_b}(X_b) \quad \lim_{z\to 0} \phi_b(z) \otimes \lambda = 0_b \ .$$

Theorem 2.9. *Let $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ such that $e^2 = \langle e, k\rangle = -1$, and let B_e be the open subset of B consisting of points b for which X_b admits a simply exceptional divisor E_b in the class e_b. If $B_e \neq \emptyset$ then*

1. *There exists an effective divisor $\mathcal{E}_0 \subset p^{-1}(B_e)$ flat over B_e whose fiber over a point $b \in B_e$ is E_b.*
2. *The map $\sigma_e : B_e \to \mathcal{P}^e$ defined by $b \mapsto [\mathcal{O}_{X_b}(E_b)]$ extends to a holomorphic section s_e of the line bundle $\mathcal{P}^e$ over B.*

(3) *One has $B_e = B \setminus Z(s_e)$, where $Z(\cdot)$ stands for the vanishing locus of a section. In particular, the complement of B_e is a divisor $H_e := Z(s_e)$, so B_e is Zariski open.*

Proof. Using Remark 2.2 and Proposition 2.3 we see that the map

$$B_e \ni b \mapsto E_b \in \mathcal{D}ou(\mathcal{X})$$

is well-defined and holomorphic. The first statement follows now from the universal property of the Douady space.

Let $(g_b)_{b\in B}$ be a smooth family of Gauduchon metrics on the fibers X_b normed such that for every $b \in B$ it holds $\deg_{g_b}(\phi_b(z)) = -\log|z|$. The map $\mathcal{P}^e \to [0,\infty)$ given by

$$\mathrm{Pic}^e(X_b) \ni [\mathcal{L}] \mapsto e^{-\deg_{g_b}(\mathcal{L})} \ , \ 0_b \mapsto 0$$

defines a norm $\|\cdot\|$ on the holomorphic line bundle $\mathcal{P}^e$ on B. We claim

Claim: For every $b_0 \in \bar{B}_e \setminus B_e$ one has $\lim_{b\to b_0} \|\sigma_e(b)\| = 0$.

Indeed, if not, there exists a sequence $(b_n)_{n\in\mathbb{N}^*}$ in B_e converging to b_0 and $\varepsilon > 0$ such that $\|\sigma_e(b_n)\| \geq \varepsilon$ for every $n \in \mathbb{N}^*$. This implies that the sequence

$$(\deg_{g_{b_n}}(\sigma_e(b_n)))_{n\in\mathbb{N}^*} = (\mathrm{vol}_{g_{b_n}}(E_{b_n}))_{n\in\mathbb{N}^*}$$

is bounded, so there exists a subsequence of $(E_{b_n})_{n\in\mathbb{N}^*}$ converging in the Barlet space $\mathcal{B}^e_B(\mathcal{X})$ [1]. By Lemma 2.8 the limit divisor E_0 in X_{b_0} will be simply exceptional, so $b_0 \in B_e$, which contradicts the choice of b_0. This proves the claim.

We define now a section $s_e : B \to \mathcal{P}^e$ by

$$s_e(b) := \begin{cases} \sigma_e(b) & \text{if} \quad b \in B_e \\ 0_b & \text{if} \quad b \notin B_e \end{cases} .$$

Using the claim proved above we see that s_e is continuous. The first statement of the theorem follows now from Rado's theorem, and the second follows from the first and the explicit construction of the section s_e. □

Remark 2.10. *The functor which associates to a pair $(p : \mathcal{X} \to B, e)$ consisting of a holomorphic family of class VII surfaces and a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ with*

$$e^2 = \langle e, k\rangle = -1$$

the (possibly empty) effective divisor $H_e \subset B$, commutes with base change. Therefore this functor should be interpreted as an effective Cartier divisor in the moduli stack classifying pairs (X, η) consisting of a class VII surface X and a class $\eta \in H_2(X, \mathbb{Z})$ with $\eta^2 = \langle \eta, \mathcal{K}_X\rangle = -1$.

Remark 2.11. *The underlying codimension 1 analytic set $\mathfrak{H}_e \subset B$ associated with H_e coincides with the hypersurface defined by Dloussky in* [5].

3. Extension theorems

Let $p : \mathcal{X} \to B$ be a holomorphic family of class VII surfaces. Using the notations and the definitions introduced in the previous section, fix a section $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2)$ such that $e^2 = \langle e, k\rangle = -1$, and let $B_e := B \setminus H_e \subset B$ the associated Zariski open subset (see Theorem 2.9). We know that H_e is the vanishing divisor of a holomorphic section $s_e \in H^0(B, \mathcal{P}^e)$, which on B_e is given by $b \mapsto [\mathcal{O}_{X_b}(E_b)]$, E_b being the unique simply exceptional divisor on X_b representing the class e_b.

As we explained in the introduction the fundamental problem studied in this article is the evolution of the divisor E_b as b tends to a point $b_0 \in H_e$. Since the volume of E_b with respect to a smooth family of Gauduchon metrics on the fibers tends to infinity

as $b \to b_0$, we cannot expect to understand this evolution using standard complex geometric tools applied to the given family.

In this section we will prove that, under certain assumptions, a lift $\widetilde{E}_b$ of E_b in the universal cover $\widetilde{X}_b$ of X_b does have a limit as $b \to b_0$, and this limit is a non-compact divisor of $\widetilde{X}_{b_0}$. Moreover, the family $(\widetilde{E}_b)_{b\in B}$ extends to a flat family of divisors over the whole base B. We can prove this type of extensions results under two types of assumptions. First we will assume that all the fibers X_b are class VII surfaces with the topological type of class VII surfaces having a non-separating strictly pseudo-convex embedded 3-sphere; second we will suppose that the subset $B_0 \subset B$ consisting of points b for which X_b is an unknown class VII surface is contained in an analytic subset of codimension ≥ 2.

The diffeomorphism type of a GSS surface with $b_2 = n$ is $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$, so if the GSS conjecture was true this would be the diffeomorphism type of *any* minimal class VII surface with $b_2 = n > 0$. Moreover, by a result of Nakamura [9], any minimal class VII surface with $b_2 = n > 0$ containing a cycle of curves is a degeneration ("big deformation") of a family of blown up primary Hopf surfaces, so it also has the diffeomorphism type of a GSS surface. By the results of [14], [15] every minimal class VII surface with $b_2 \in \{1, 2\}$ does contain a cycle, so by Nakamura's result, it has the same diffeomorphism type. Therefore, studying class VII surfaces with the diffeomorphism (or, more generally, homeomorphism) type $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$, and studying deformations of such surfaces, is an important problem.

In general, if M is a topological 4-manifold M with this topological type, then every 2-homology class $e \in H_2(M, \mathbb{Z})$ is represented by an embedded surface with simply connected components, in particular it lifts to $H_2(\widetilde{M}, \mathbb{Z})$. More precisely, $H_2(\widetilde{M}, \mathbb{Z})$ is naturally an n-dimensional free $\mathbb{Z}[F]$-module, where F stands for the multiplicative group $\mathrm{Aut}_M(\widetilde{M}) = \{f^k |\ k \in \mathbb{Z}\} \simeq \mathbb{Z}$, and the quotient

$$H_2(\widetilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}[F]} \mathbb{Z} = {}^{H_2(\widetilde{M}, \mathbb{Z})}\!/_{(\mathrm{id} - f)H_2(\widetilde{M}, \mathbb{Z})}$$

is naturally isomorphic to $H_2(M, \mathbb{Z})$. Here $\mathbb{Z}$ is regarded as a $\mathbb{Z}[F]$-algebra via the trace morphism

$$\sum_{n\in\mathbb{Z}} a_n f^n \mapsto \sum_n a_n \ ,$$

and $(\mathrm{id} - f)$ stands for the ideal $(\mathrm{id} - f) = (\mathrm{id} - f)\mathbb{Z}[F] \subset \mathbb{Z}[F]$ generated by $\mathrm{id} - f$.

Let $p : \mathcal{M} \to B$ be a locally trivial topological fiber bundle with $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$ as fiber and 2-connected basis B, $\alpha : \widetilde{\mathcal{M}} \to \mathcal{M}$ the universal cover of the total space, and $\widetilde{p} := p \circ \alpha : \widetilde{\mathcal{M}} \to B$ the induced fiber bundle. Since B is 2-connected, the homotopy long exact sequence associated with the locally trivial fiber bundle p shows that the fiber embedding $\iota_b : M_b \hookrightarrow \mathcal{M}$ induces an isomorphism $\pi_1(M_b, m) \xrightarrow{\simeq} \pi_1(\mathcal{M}, m)$, so the induced projection $\alpha_b : \widetilde{M}_b \to M_b$ is a universal cover of the fiber M_b for every $m \in M_b$. Note also that the assumption "B is 2-connected" implies that the locally constant sheaves $\underline{H}_i$, $\underline{H}^i$ on B are constant, for every $i \in \mathbb{N}$. Denoting by $\underline{\widetilde{H}}_i$, $\underline{\widetilde{H}}^i$ the corresponding sheaves associated with the fiber bundle $\widetilde{p}$ we obtain sheaf epimorphisms $\underline{\widetilde{H}}_i \to \underline{H}_i$ and sheaf monomorphisms $\underline{H}^i \to \underline{\widetilde{H}}^i$ defined by the morphism sequences $(\alpha_{b*})_{b\in B}$, $(\alpha_b^*)_{b\in B}$.

We denote by f a generator of $\mathrm{Aut}_{\mathcal{M}}(\widetilde{\mathcal{M}})$ and by f_b the induced generator of $\mathrm{Aut}_{M_b}(\widetilde{M}_b)$. With these notations we obtain, for any $b \in B$, a commutative diagram

with horizontal short exact sequences:

$$\begin{array}{ccccccc} 0 \to & (\mathrm{id}-f_b)H_2(\widetilde{M}_b,\mathbb{Z}) & \hookrightarrow & H_2(\widetilde{M}_b,\mathbb{Z}) & \xrightarrow{\alpha_{b,*}} & H_2(M_b,\mathbb{Z}) & \to 0 \\ & \uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq & \\ 0 \to & (\mathrm{id}-f)H^0(B,\widetilde{\underline{H}}_2) & \hookrightarrow & H^0(B,\widetilde{\underline{H}}_2) & \xrightarrow{\alpha_*} & H^0(B,\underline{H}_2) & \to 0 \end{array} .$$

Let X be a class VII surface with topological type $(S^1\times S^3)\#n\bar{\mathbb{P}}^2$ and Σ a strictly pseudo-convex non-separating embedded 3-sphere in X. A lift $\widetilde{\Sigma}$ of Σ to $\widetilde{X}$ separates $\widetilde{X}$ in two manifolds with common boundary $\widetilde{\Sigma}$, so we can define the pseudo-convex and pseudo-concave ends of $\widetilde{X}$ as in the case of GSS surfaces (see section 1.2): the pseudo-convex end of $\widetilde{X}$ is the end defined by the connected component of $\widetilde{X}\setminus\widetilde{\Sigma}$ whose closure has strictly pseudo-concave boundary. One can prove that this labeling of the ends is coherent, i.e. independent of the choice of Σ and $\widetilde{\Sigma}$.

Theorem 3.1. *(first extension theorem) Let $p:\mathcal{X}\to B$ be a holomorphic family of class VII surfaces with 2-connected basis B, such that every fiber X_b*

1. *has the topological type $(S^1\times S^3)\#n\bar{\mathbb{P}}^2$,*
2. *admits a non-separating strictly pseudo-convex embedded 3-sphere Σ_b.*

Let $e=(e_b)_{b\in B}\in H^0(B,\underline{H}_2)$ with $e^2=\langle e,k\rangle=-1$ such that $B_e\neq\emptyset$. Let $\alpha:\widetilde{\mathcal{X}}\to\mathcal{X}$ be a universal cover of $\mathcal{X}$, and $\widetilde{e}=(\widetilde{e}_b)_{b\in B}$ a lift of e in $H^0(B,\widetilde{\underline{H}}_2)$. Then there exists an effective divisor $\widetilde{\mathcal{E}}\subset\widetilde{\mathcal{X}}$ with the following properties:

1. *For any point $b\in B_e$, the fiber $\widetilde{E}_b:=\widetilde{\mathcal{E}}\cap\widetilde{X}_b$ is a lift representing $\widetilde{e}_b$ of the unique simply exceptional divisor E_b in the class e_b,*
2. *For any point $b\in H_e$, the fiber $\widetilde{E}_b$ is bounded towards the pseudo-convex end of $\widetilde{X}_b$, but unbounded towards its pseudo-concave end.*
3. *$\widetilde{\mathcal{E}}$ is flat over B.*

Proof. Using Theorem 2.9 we get an effective divisor $\mathcal{E}_0\subset p^{-1}(B_e)$, flat over B_e, whose fiber over a point $b\in B_e$ is the unique simply exceptional divisor $E_b\subset X_b$ in the class e_b. Since E_b is a tree of rational curves (so simply connected), it can be lifted to $\widetilde{X}_b$, and the set of lifts can be identified with the set $\alpha_{b,*}^{-1}(e_b)\subset H_2(\widetilde{X}_b,\mathbb{Z})$. Moreover, since E_b has a simply connected neighborhood in $\mathcal{X}$, such a lift can be locally chosen such that it depends holomorphically on $b\in B_e$.

For $b\in B_e$ let $\widetilde{E}_b$ be the lift of E_b representing the given class $\widetilde{e}_b\in H_2(\widetilde{X}_b,\mathbb{Z})$. Using Proposition 2.3 we see that map $B_e\ni b\mapsto\widetilde{E}_b\in\mathcal{D}ou^e(\mathcal{X})$ is biholomorphic. The union

$$\widetilde{\mathcal{E}}_0:=\bigcup_{b\in B_e}\widetilde{E}_b$$

is an effective divisor of $\widetilde{p}^{-1}(B_e)\subset\widetilde{\mathcal{X}}$, and the restriction $\alpha_{|\widetilde{\mathcal{E}}_0}:\widetilde{\mathcal{E}}_0\to\mathcal{E}_0$ is biholomorphic.

The idea of the proof is to prove, using the Remmert-Stein theorem, that the closure $\widetilde{\mathcal{E}}:=\bar{\widetilde{\mathcal{E}}}_0$ of $\widetilde{\mathcal{E}}_0$ in $\widetilde{\mathcal{X}}$ is a divisor, and that this divisor has the desired properties. The irreducible components of $\widetilde{H}_e:=\widetilde{\mathcal{X}}\setminus\widetilde{p}^{-1}(B_e)=\widetilde{p}^{-1}(H_e)$ have the form $\widetilde{p}^{-1}(H)$, where H is an irreducible component of the divisor H_e (see Theorem 2.9). Let H_0 be such a component and $b_0\in H_0$. Using the second property of the fibers, we get an embedding $s_0:S^3\to X_{b_0}$ whose image Σ_0 is non-separating and strictly pseudo-convex. The image $\widetilde{\Sigma}_0:=\mathrm{im}(\widetilde{s}_0)$ of a lift $\widetilde{s}_0:S^3\to\widetilde{X}_{b_0}$ of this embedding to $\widetilde{X}_{b_0}$ separates $\widetilde{X}_{b_0}$ in two manifolds $\widetilde{X}_{b_0}^{\pm}$, the first with strictly pseudo-concave, the second with strictly pseudo-convex boundary $\widetilde{\Sigma}_0$.

We can deform the lift $\widetilde{s}_0$ to get a smooth family of embeddings $(\widetilde{s}_b:S^3\to\widetilde{\mathcal{X}})_{b\in U}$, where U is a connected open neighborhood of b_0 in B, such that $\widetilde{s}_{b_0}=\widetilde{s}_0$, and for every

$b \in U$ the image $\widetilde{\Sigma}_b := \mathrm{im}(\widetilde{s}_b)$ is a strictly pseudo-convex separating hypersurface of $\widetilde{X}_b$. Therefore, for every $b \in U$ the universal cover $\widetilde{X}_b$ of X_b decomposes as the union of two manifolds $\widetilde{X}_b^{\pm}$, the first with pseudo-concave, the second with pseudo-convex boundary $\widetilde{\Sigma}_b$. Similarly, $\widetilde{\mathcal{X}}_U := \widetilde{p}^{-1}(U)$ decomposes as the union of two manifolds $\widetilde{\mathcal{X}}_U^{\pm} = \cup_{b\in U}\widetilde{X}_b^{\pm}$ with common boundary $\widetilde{\mathfrak{S}}_U := \cup_{b\in U}\widetilde{\Sigma}_b$. We choose the generator f of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ such that the induced automorphism f_b of $\widetilde{X}_b$ maps $\widetilde{X}_b^-$ into itself (so it moves the points towards the pseudo-concave end) for every $b \in U$.

Choose a point $v_0 \in B_e \cap U$. Since the divisor $\widetilde{E}_{v_0} \subset \widetilde{X}_{v_0}$ is compact, we can find $m \in \mathbb{N}$ sufficiently large such that $(f_{v_0}^{-m}(\widetilde{X}_{v_0}^+)) \cap \widetilde{E}_{v_0} = \emptyset$. We claim

Claim: $f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{E}}_0 = \emptyset$.

Indeed, let $V \subset U \cap B_e$ the subset of points v for which $(f_v^{-m}(\widetilde{X}_v^+)) \cap \widetilde{E}_v = \emptyset$. Since all divisors $\widetilde{E}_b$, $b \in B_e$ are compact, it follows easily that V is open in $U \cap B_e$. We will show that it is also closed in $U \cap B_e$; in order to check this, let $(v_n)_{n\in\mathbb{N}^*}$ be a sequence in V converging to a point $w \in U \cap B_e$. Since $\widetilde{E}_{v_n} \subset f_{v_n}^{-m}(\widetilde{X}_{v_n}^-)$ for every $n \in \mathbb{N}^*$ it follows that $\widetilde{E}_w \subset f_w^{-m}(\widetilde{X}_w^-)$. The intersection $\widetilde{E}_w \cap f_w^{-m}(\widetilde{\Sigma}_w)$ is empty, because $f_w^{-m}(\widetilde{\Sigma}_w)$ regarded as boundary of $f_w^{-m}(\widetilde{X}_w^-)$ is strictly pseudo-convex. Therefore $\widetilde{E}_w \cap f_w^{-m}(\widetilde{X}_w^+) = \emptyset$, so $w \in V$.

Therefore V is both open and closed in $U \cap B_e = U \setminus H_e$, which is connected, so $V = U \cap B_e$. Therefore $(f_u^{-m}(\widetilde{X}_u^+)) \cap \widetilde{\mathcal{E}}_0 = \emptyset$ for every $u \in U \cap B_e$, so for every $u \in U$; taking into account that $\bigcup_{u\in U} f_u^{-m}(\widetilde{X}_u^+) = f^{-m}(\widetilde{\mathcal{X}}_U^+)$, this proves the claim.

Note now that $f^{-m}(\widetilde{\mathcal{X}}_U^+)$ is a neighborhood of any point in $f_{b_0}^{-m-1}(\widetilde{X}_{b_0}^+)$. Therefore the claim implies that $\widetilde{H}_0$ contains points which do not belong to the closure $\overline{\widetilde{\mathcal{E}}}_0$ of the divisor $\widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{X}} \setminus \widetilde{H}_e$. Since this holds for every irreducible component $\widetilde{H}_0$ of $\widetilde{H}_e$, the existence of a divisor $\mathcal{E}$ satisfying the first property stated in the theorem follows now from the Remmert-Stein theorem (see [13] Korollar zu Satz 12, p. 300). For the second property note first that for any $b_0 \in H_e$ we have $(f_{b_0}^{-m}(\widetilde{X}_{b_0}^+)) \cap \widetilde{E}_{b_0} = \emptyset$, so $\widetilde{E}_{b_0}$ is indeed bounded towards the pseudo-convex end of $\widetilde{X}_{b_0}$. On the other hand $\widetilde{E}_{b_0}$ cannot be compact because, if it were, its projection on X_{b_0} would be a limit of simply exceptional divisors, so itself simply exceptional (contradicting $b_0 \in H_e$).

For the third property note that for every $b \in B$ the fiber $\widetilde{E}_b$ of $\widetilde{\mathcal{E}}$ is a divisor of $\widetilde{X}_b$, so the result follows from Lemma 2.4.

□

Remark 3.2. *Note that for $b \in B_e$ the divisor $\widetilde{E}_b$ represents the class $\widetilde{e}_b \in H_2(\widetilde{X}_b, \mathbb{Z})$. This property cannot be extended for $b \in H_e$ because in this case $\widetilde{E}_b$ is no longer compact. However, using the flatness property of $\widetilde{\mathcal{E}}$ over B we see that for every $b \in B$*

1. *$c_1(\mathcal{O}_{X_b}(\widetilde{E}_b))$ is the image of the Poincaré dual $PD(\widetilde{e}_b) \in H_c^2(\widetilde{X}_b, \mathbb{Z})$ of $\widetilde{e}_b$ via the natural morphism $H_c^2(\widetilde{X}_b, \mathbb{Z}) \to H^2(\widetilde{X}_b, \mathbb{Z})$.*
2. *$\widetilde{E}_b$ represents the image of $\widetilde{e}_b$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(\widetilde{X}_b, \mathbb{Z})$.*

Taking into account Remark 3.2 and Theorem 3.1 we obtain immediately Theorem 1.2 stated in the introduction.

Remark 3.3. *The assignment $(p : \mathcal{X} \to B, \alpha : \widetilde{\mathcal{X}} \to \mathcal{X}, \widetilde{e}) \mapsto \widetilde{\mathcal{E}} \subset \widetilde{\mathcal{X}}$ given by the proof of Theorem 3.1 is compatible with base changes $\chi : B' \to B$ satisfying $\chi^{-1}(B_e) \neq \emptyset$.*

Using this remark one can prove that the assumption $B_e \neq \emptyset$ in Theorem 3.1 can be replaced with one of the following equivalent assumptions:

(i) *For every $b \in B$ there exists small deformations of X_b admitting a simply exceptional divisors in the class e_b.*

(ii) *For every $b \in B$ the versal deformation of X_b contains surfaces admitting a simply exceptional divisors in the class e_b.*

In particular, the theorem holds when all the fibers are GSS surfaces, because any GSS surface can be deformed into simply blown up primary Hopf surfaces. We will see in the next section that for families of GSS surfaces, the divisors $\widetilde{E}_b$, $b \in H_e$ are always given by infinite series of compact curves.

Using Remark 3.3 we obtain Theorem 1.3 stated in the introduction.

Coming back to the conditions and the proof of Theorem 3.1, consider the semigroup $G_+ := \{f^n |\ n \in \mathbb{N}\}$ of transformations of $\widetilde{\mathcal{X}}$ which act fiberwise with respect to the fibration $\widetilde{p} : \widetilde{\mathcal{X}} \to B$, where the generator f of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ is chosen again such that it moves the points towards the pseudo-concave ends of the fibers $\widetilde{X}_b$.

Remark 3.4. *The closure $\overline{\widetilde{\mathcal{F}}_0}$ of the effective divisor*

$$\widetilde{\mathcal{F}}_0 := G_+(\widetilde{\mathcal{E}}_0) = \bigcup_{n\in\mathbb{N}} f^n(\widetilde{\mathcal{E}}_0) \subset \widetilde{p}^{-1}(B_e)$$

in $\widetilde{\mathcal{X}}$ is also an effective divisor of $\widetilde{\mathcal{X}}$, flat over B.

Proof. We use the same arguments as in the proof Theorem 3.1 and we take into account that G_+ moves $\widetilde{\mathcal{E}}_0$ towards the pseudo-concave end. More precisely, the property $f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{E}}_0 = \emptyset$ established in the proof implies $f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{F}}_0 = \emptyset$. Indeed, if for a non-negative integer n one had $x \in f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap f^n(\widetilde{\mathcal{E}}_0)$, then

$$f^{-n}(x) \in f^{-m-n}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{E}}_0 \subset f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{E}}_0\ ,$$

so the intersection $f^{-m}(\widetilde{\mathcal{X}}_U^+) \cap \widetilde{\mathcal{E}}_0$ would be non-empty. Here we used the obvious inclusion $f^{-n}(\widetilde{\mathcal{X}}_U^+) \subset \widetilde{\mathcal{X}}_U^+$. □

Note that the similar statement formulated for the semigroup $G_- := \{f^{-n} |\ n \in \mathbb{N}\}$ is not true in general.

Our second extension theorem concerns families of class VII surfaces for which the existence of strictly pseudo-convex embedded 3-spheres is assumed only for the fibers X_b, $b \in B \setminus A$, where A is an analytic subset of codimension ≥ 2 of B and contained in the divisor H_e. This theorem contains obviously Theorem 1.4 stated in the introduction as a special case.

The ends of the universal covers $\widetilde{X}_b$ of the fibers form a trivial double cover of B, so one can speak about the pseudo-convex and the pseudo-concave end of X_b even for points $b \in A$.

Theorem 3.5. *(second extension theorem) Let $p : \mathcal{X} \to B$ be a holomorphic family of class VII surfaces with 2-connected basis B. Let $e = (e_b)_{b\in B} \in H^0(B, \underline{H}_2) = H_2(\mathcal{X}, \mathbb{Z})$ with $e^2 = \langle e, k\rangle = -1$ such that $B_e \neq \emptyset$, and A an analytic subset of codimension ≥ 2 of B such that $A \subset H_e$ and, for every $b \in H_e \setminus A$, the fiber X_b has the properties*

(1) *has the topological type $(S^1 \times S^3)\#n\bar{\mathbb{P}}^2$,*
(2) *admits a non-separating strictly pseudo-convex embedded 3-sphere Σ_b.*

Let $\alpha : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a universal cover of $\mathcal{X}$, and $\widetilde{e} = (\widetilde{e}_b)_{b\in B}$ a lift of e in $H^0(B, \underline{\widetilde{H}}_2)$. Then there exists an effective divisor $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{X}}$ with the following properties:

(i) *For any point $b \in B_e$, the fiber $\widetilde{E}_b := \widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is a lift of the unique simply exceptional divisor E_b in the class e_b. This lift represents the class $\widetilde{e}_b$.*
(ii) *for any point $b \in H_e = B \setminus B_e$, the fiber $\widetilde{E}_b$ is a divisor of $\widetilde{X}_b$ which is bounded towards the pseudo-convex end of $\widetilde{X}_b$, but unbounded towards its pseudo-concave end.*

(iii) $\widetilde{\mathcal{E}}$ *is flat over* B.

Proof. Using Theorem 2.9 again we obtain a divisor $\mathcal{E}_0 \subset p^{-1}(B_e)$, flat over B_e, whose fiber over a point $b \in B_e$ is the unique simply exceptional divisor $E_b \subset X_b$ in the class e_b. Applying our first extension theorem Theorem 3.1 to the family $\widetilde{p}^{-1}(B \setminus A) \to B \setminus A$ we get an effective divisor $\widetilde{\mathcal{E}}^0 \subset \widetilde{p}^{-1}(B \setminus A)$ flat over $B \setminus A$ satisfying the property (i) and also the property (ii) for any point $b \in B \setminus A$. So $\widetilde{\mathcal{E}}^0$ is the closure in $\widetilde{p}^{-1}(B \setminus A)$ of the lift $\widetilde{\mathcal{E}}_0 \subset \widetilde{p}^{-1}(B \setminus H_e)$ of $\mathcal{E}_0$ obtained by lifting fiberwise the simply exceptional divisors E_b in the homology class $\widetilde{e}_b$ (for $b \in B_e$).

Since $\widetilde{p}^{-1}(A)$ has codimension ≥ 2 in $\widetilde{\mathcal{X}}$, the closure

$$\widetilde{\mathcal{E}} := \overline{\widetilde{\mathcal{E}}^0} = \overline{\widetilde{\mathcal{E}}_0}$$

of $\widetilde{\mathcal{E}}^0$ in $\widetilde{\mathcal{X}}$ is an effective divisor of $\widetilde{\mathcal{X}}$ by the second Remmert-Stein extension theorem (see [13] Satz 13 p. 299), where $\overline{\ }$ denotes everywhere closure in $\widetilde{\mathcal{X}}$. With this choice the first statement of the theorem is proved.

For the second statement, the first problem is to show that for $b \in A$ the intersection $\widetilde{E}_b := \widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is a divisor of $\widetilde{X}_b$, i.e. that $\widetilde{E}_b$ does not coincide with the whole surface $\widetilde{X}_b$. We will show first that $\widetilde{\mathcal{E}} \cap \widetilde{X}_b$ is bounded towards the pseudo-convex end, which will imply that $\widetilde{E}_b \cap \widetilde{X}_b \neq \widetilde{X}_b$, so this intersection is indeed a divisor of $\widetilde{X}_b$ with the desired property.

Consider the semigroup $G_+ := \{f^n |\ n \in \mathbb{N}\}$ of transformations of $\widetilde{\mathcal{X}}$ which act fiberwise with respect to the fibration $\widetilde{p} : \widetilde{\mathcal{X}} \to B$. Here we choose the generator f of $\mathrm{Aut}_{\mathcal{X}}(\widetilde{\mathcal{X}})$ as in the proof of Theorem 3.1, so that it moves the points towards the pseudo-concave ends of the fibers $\widetilde{X}_b$.

Using Remark 3.4 we see that the closure $\widetilde{\mathcal{F}}^0$ of the effective divisor

$$\widetilde{\mathcal{F}}_0 := G_+(\widetilde{\mathcal{E}}_0) = \bigcup_{n \in \mathbb{N}} f^n(\widetilde{\mathcal{E}}_0) \subset \widetilde{p}^{-1}(B_e)$$

in $\widetilde{p}^{-1}(B \setminus A)$ is also an effective divisor of $\widetilde{p}^{-1}(B \setminus A)$, flat over $B \setminus A$. By the second Remmert-Stein extension theorem cited above we get an effective divisor

$$\widetilde{\mathcal{F}} = \overline{\widetilde{\mathcal{F}}^0} = \overline{\widetilde{\mathcal{F}}_0}$$

in $\widetilde{\mathcal{X}}$, where again $\overline{\ }$ denotes everywhere closure in $\widetilde{\mathcal{X}}$. For every $n \in \mathbb{N}$, we have $f^n(\widetilde{\mathcal{E}}_0) \subset \widetilde{\mathcal{F}}$ so, since the right hand space is closed, we get $f^n(\widetilde{\mathcal{E}}) \subset \widetilde{\mathcal{F}}$, which proves the inclusion

$$\forall n \in \mathbb{N}\ , f^n(\widetilde{\mathcal{E}}) \subset \widetilde{\mathcal{F}}\ . \tag{4}$$

Recall that the effective divisor $\widetilde{\mathcal{E}}_0$ of $\widetilde{p}^{-1}(B_e)$ is identified with $\mathcal{E}_0$ via the covering map α. Since the set of irreducible components of $\mathcal{E}_0$ is finite (bounded by the minimal number of irreducible components of the simply exceptional divisors E_b as b varies in B_e), the same will hold for $\widetilde{\mathcal{E}}_0$. Let $\widetilde{\mathcal{C}}_0$ be an irreducible component of $\widetilde{\mathcal{E}}_0$. Using the two Remmert-Stein theorems cited above, we see that the closure $\widetilde{\mathcal{C}}^0$ of $\widetilde{\mathcal{C}}_0$ in $\widetilde{p}^{-1}(B \setminus A)$ is an effective divisor in this manifold, and that $\widetilde{\mathcal{C}} := \overline{\widetilde{\mathcal{C}}^0} = \overline{\widetilde{\mathcal{C}}_0}$ is an effective divisor of $\widetilde{\mathcal{X}}$. We will show that

Claim 1: $\widetilde{\mathcal{C}} \cap \widetilde{X}_b$ *is bounded towards the pseudo-convex end of* $\widetilde{X}_b$ *for every* $b \in A$.

Since $\widetilde{\mathcal{E}}$ is the (finite!) union of the closures $\widetilde{\mathcal{C}}$ of the irreducible components $\widetilde{\mathcal{C}}_0$ of $\widetilde{\mathcal{E}}_0$, the claim implies obviously the second statement of the theorem.

We prove now Claim 1. Choose a point $b_0 \in A$. Since p is a locally trivial differentiable fiber bundle, it follows that there exists a smooth embedding $s_0 : S^3 \to X_{b_0}$

whose image Σ_0 does not disconnect X, and let $\widetilde{s}_0 : S^3 \to \widetilde{X}_{b_0}$ be a lift of s_0 to $\widetilde{X}_{b_0}$. We can deform $\widetilde{s}_0$ to get a smooth family of embeddings $(\widetilde{s}_b : S^3 \to \widetilde{\mathcal{X}})_{b\in U}$, where U is a connected open neighborhood of b_0 in B, such that $\widetilde{s}_{b_0} = \widetilde{s}_0$ and s_b is an embedding of S^3 in $\widetilde{X}_b$ for every $b \in U$. We can regard this family as an embedding $\widetilde{s} : U \times S^3 \to \widetilde{\mathcal{X}}_U := \widetilde{p}^{-1}(U)$, whose image $\widetilde{\Sigma}_U$ separates $\widetilde{\mathcal{X}}_U$ in two connected components. Moreover, $\widetilde{\mathcal{X}}_U$ can be written as an infinite union

$$\widetilde{\mathcal{X}}_U = \bigcup_{n\in\mathbb{Z}} \widetilde{\mathcal{X}}_U^n \ ,$$

where $\widetilde{\mathcal{X}}_U^n = f^n(\widetilde{\mathcal{X}}_U^0)$ is a manifold with two boundary components $f^n(\widetilde{\Sigma}_U)$, $f^{n+1}(\widetilde{\Sigma}_U)$.

Let $V \Subset U$ an open neighborhood of b_0 with compact closure $\bar{V} \subset U$, and put

$$\widetilde{\mathcal{X}}_{\bar{V}} := \widetilde{p}^{-1}(\bar{V}) \ , \ \widetilde{\Sigma}_{\bar{V}} := \widetilde{\Sigma}_U \cap \widetilde{\mathcal{X}}_{\bar{V}} \ , \ \widetilde{\mathcal{X}}_{\bar{V}}^n := \widetilde{\mathcal{X}}_U^n \cap \widetilde{\mathcal{X}}_{\bar{V}} \ ,$$

and note that the spaces $\widetilde{\mathcal{X}}_{\bar{V}}^n = f^n(\widetilde{\mathcal{X}}_{\bar{V}}^0)$ are all compact. Our claim follows from

Claim 2: $\widetilde{\mathcal{C}} \cap \widetilde{\mathcal{X}}_{\bar{V}}$ is bounded towards the pseudo-convex end.

Indeed if, by reductio ad absurdum, $\widetilde{\mathcal{C}} \cap \widetilde{\mathcal{X}}_{\bar{V}}$ were not bounded towards the pseudo-convex end, then the same would be true for $f^n(\widetilde{\mathcal{C}}) \cap \widetilde{\mathcal{X}}_{\bar{V}}$ for every $n \geq 0$. Therefore the irreducible divisors $f^n(\widetilde{\mathcal{C}})$ intersect the compact subset $\widetilde{\mathcal{X}}_{\bar{V}}^0$ of $\widetilde{\mathcal{X}}$ for all sufficiently large $n \geq 0$.

On the other hand using the inclusion $\cup_{n\geq 0} f^n(\widetilde{\mathcal{C}}) \subset \widetilde{\mathcal{F}}$ given by (4), we see that $\cup_{n\geq 0} f^n(\widetilde{\mathcal{C}})$ is a divisor. With this remark we can apply Lemma 3.6 below, according to which it would follow that the sequence $(f^n(\widetilde{\mathcal{C}}))_{n\geq 0}$ is finite, which is of course impossible, because the intersections of these divisors with $\widetilde{p}^{-1}(B_e)$ are pairwise different.

The third statement of the theorem follows now using the same arguments as in the proof of the similar statement in Theorem 3.1. □

Lemma 3.6. *Let M be a connected complex manifold, and let $(D_n)_{n\in\mathbb{N}}$ be a sequence of effective irreducible divisors in M. Suppose that*

1. *The union $D := \bigcup_{n\in\mathbb{N}} D_n$ is a divisor of M, and*
2. *there exists a compact subspace $K \subset M$ such that $K \cap D_n \neq \emptyset$ for every $n \in \mathbb{N}$.*

Then the set of divisors $\{D_n |\ n \in \mathbb{N}\}$ is finite.

Proof. If not, there would exist a subsequence $(D_{n_m})_{m\in\mathbb{N}}$ of $(D_n)_{n\in\mathbb{N}}$ such that $D_{n_k} \neq D_{n_l}$ for every $k \neq l$. Since all these divisors are irreducible, it follows that:

Remark: For $k \neq l$, the analytic set $D_{n_k} \cap D_{n_l}$ does not contain any codimension 1 irreducible component.

For every $m \in \mathbb{N}$, let $x_m \in K \cap D_{n_m}$, and let $x \in K$ be the limit of a convergent subsequence $(x_{m_s})_{s\in\mathbb{N}}$ of $(x_m)_{m\in\mathbb{N}}$. Let U be a connected open neighborhood of x, so U contains all the points x_{m_s} with $s \geq s_0$ for a sufficiently large index $s_0 \in \mathbb{N}$. For every $s \geq s_0$ we consider an irreducible component E_s of the intersection $D_{n_{m_s}} \cap U$, so E_s is a *non-empty* effective divisor of U. Taking into account the remark above, it follows that $E_s \neq E_t$ for $s \neq t$.

Therefore, the intersection $D \cap U = \left(\bigcup_{n\in\mathbb{N}} D_n\right) \cap U$ contains the infinite union $\bigcup_{s\geq s_0} E_s$ of pairwise distinct irreducible effective divisors E_s, so $D \cap U$ cannot be a divisor of U. This contradicts the first assumption of our hypothesis.

□

Using the flatness of the obtained divisor $\widetilde{\mathcal{E}}$ over B we see as in Remark 3.2 that, in the conditions of Theorem 3.5

Remark 3.7. *For every point $b \in B$ one has*

(1) *$c_1(\mathcal{O}_{X_b}(\widetilde{E}_b))$ is the image of the Poincaré dual $PD(\widetilde{e}_b) \in H^2_c(\widetilde{X}_b, \mathbb{Z})$ of $\widetilde{e}_b$ via the natural morphism $H^2_c(\widetilde{X}_b, \mathbb{Z}) \to H^2(\widetilde{X}_b, \mathbb{Z})$.*

(2) *$\widetilde{E}_b$ represents the image of $\widetilde{e}_b$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(\widetilde{X}_b, \mathbb{Z})$.*

A natural problem concerning the two extensions theorems Theorem 3.1 and 3.5 is to understand, for $b \in H_e$, the dependence of the obtained non-compact divisor $\widetilde{E}_b \subset \widetilde{X}_b$ on the lift $\widetilde{e}_b$ of e_b. It is easy to see that

Remark 3.8. *For a fixed class e_b with $e_b^2 = \langle e_b, k_b \rangle = -1$, the divisors $\widetilde{E}_b$, $\widetilde{E}'_b$ associated with two different lifts $\widetilde{e}_b$, $\widetilde{e}'_b$ of e_b are mapped on each other by a power f^n of the generator f of the automorphism group $\mathrm{Aut}_{X_b}(\widetilde{X}_b)$.*

When the fiber X_b is a GSS surface much more can be said: in the next section we shall see that, if $X := X_b$ is a GSS surface, then the effective divisors $\widetilde{E} := \widetilde{E}_b \subset \widetilde{X}$ are always given by series of *compact* curves. Moreover, in this case we shall see that there exists a natural *total* order in the set $\{\widetilde{e} \in H_2(\widetilde{X}, \mathbb{Z}) |\ \alpha_*(\widetilde{e})^2 = \langle \alpha_*(\widetilde{e}), k_b \rangle = -1\}$, and endowed with this order, this set (which is a basis of $H_2(\widetilde{X}, \mathbb{Z})$) can be identified with $(\mathbb{Z}, \leq)$. The map $\widetilde{e} \mapsto \widetilde{E}$ which associates to such a class the corresponding effective divisor, is monotone decreasing, in the sense that $\widetilde{E} \leq \widetilde{E}'$ whenever $\widetilde{e}' \leq \widetilde{e}$ (see Proposition 4.6, Remark 4.7). At this moment we cannot prove these properties for a fiber X_b ($b \in H_e$) which is not assumed to be a GSS surface.

4. The case of GSS surfaces. The fundamental series

We begin with the following

Definition 4.1. *Let X be a compact complex surface. We say that X contains a global spherical shell (GSS), or that X is a GSS surface, if there exists a biholomorphic map $\varphi : U \to X$ from a neighborhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere S^3 into X such that $X \setminus \varphi(S^3)$ is connected.*

Primary Hopf surfaces are the simplest examples of GSS surfaces. The differential topological type of a GSS surface X with $b_2(X) = n$ is $(S^1 \times S^3) \# n\overline{\mathbb{P}}^2$, so $H_2(X, \mathbb{Z})$ admits an (unordered) basis $\mathfrak{B} \subset H_2(X, \mathbb{Z})$ trivializing the intersection form q_X of X, i.e. such that

$$e' \cdot e'' = \begin{cases} 0 & \text{if} \quad e' \neq e'' \\ -1 & \text{if} \quad e' = e'' \end{cases} .$$

Decomposing the Chern class $c_1(\mathcal{K}_X) = -c_1(X) \in H^2(X, \mathbb{Z})$ with respect to the Poincaré dual basis $\mathfrak{B}^\vee = \{PD(e) |\ e \in \mathfrak{B}\}$ of $H^2(X, \mathbb{Z})$ we get

$$c_1(\mathcal{K}_X) = \sum_{e \in \mathfrak{B}} k_e PD(e)$$

with $k_e \equiv 1 \bmod 2$ (because $c_1(\mathcal{K}_X)$ is a characteristic element) and $\sum_{e \in \mathfrak{B}} k_e^2 = -c_1(X)^2 = c_2(X) = n$. This shows that $k_e \in \{\pm 1\}$ so, replacing some of the elements $e \in \mathfrak{B}$ by $-e$ if necessary, we may assume that

$$c_1(\mathcal{K}_X) = \sum_{e \in \mathfrak{B}} PD(e) \ , \tag{5}$$

or equivalently

$$\forall e \in \mathfrak{B} \ , \ \langle e, c_1(\mathcal{K}_X) \rangle = -1 \ . \tag{6}$$

There exists a unique unordered q_X-trivializing basis $\mathfrak{B}$ with this property, and this basis will be called *the standard basis* of $H_2(X, \mathbb{Z})$. Note that the homology class of any simply exceptional divisor of X (if such a divisor exists) is an element of its standard

basis.

For surfaces X with GSS, the second Betti number $b_2(X)$ is equal to the number of rational curves in X. A *marked GSS surface* is a pair (X, C_0) consisting of GSS surface with $b_2(X) \geq 1$ and a rational curve C_0 in X.

Any minimal marked GSS surface (X, C_0) with $n = b_2(X) \geq 1$ can be obtained using a simple 2-step construction consisting of an iterated blow up followed by a holomorphic surgery. More precisely, let $\Pi = \Pi_0 \circ \cdots \circ \Pi_{n-1} : B^\Pi \to B$ be a sequence of n blowing ups such that

- the first blowing up $\Pi_0 : B^{\Pi_0} \to B$ blows up the origin $O_{-1} := (0,0)$ in the 2-dimensional unit ball $B \subset \mathbb{C}^2$,
- for $i = 0, \ldots, n-2$ the blowing up $\Pi_{i+1} : B^{\Pi_0 \circ \cdots \circ \Pi_{i+1}} \to B^{\Pi_0 \circ \cdots \circ \Pi_i}$ blows up a point $O_i \in \Pi_i^{-1}(O_{i-1})$ in the surface $B^{\Pi_0 \circ \cdots \circ \Pi_i}$ obtained at the previous step.

Applying the same sequence of blowing ups to a ball $B(r)$ of radius r and to its closure $\bar{B}(r)$ we obtain complex surfaces (respectively compact complex surfaces with boundary) which will be denoted by $B(r)^\Pi$, respectively $\bar{B}(r)^\Pi$.

Let now $\sigma : \bar{B} \to B^\Pi$ an embedding which extends to a biholomorphism from a neighborhood of $\bar{B}$ onto a small open ball in B^Πsuch that $\sigma(0) = O_{n-1} \in \Pi_{n-1}^{-1}(O_{n-2})$. One can associate to the pair (Π, σ) a minimal surface $X = X(\Pi, \sigma)$ by removing from $\bar{B}^\Pi$ the image $\sigma(B)$, and identifying the two boundary components $\partial\bar{B}^\Pi$, $\sigma(\partial\bar{B})$ of the compact manifold with boundary $A := \bar{B}^\Pi \setminus \sigma(B)$ by $\sigma \circ \Pi$, the holomorphic structure on the resulting differentiable manifold being defined such that the natural surjective locally diffeomorphic map

$$q_{\Pi,\sigma} : B(1+\epsilon)^\Pi \setminus \sigma(\bar{B}(1-\epsilon)) \longrightarrow X(\Pi, \sigma) \ .$$

is a local biholomorphism. One can prove that for any marked surface (X, C_0) there exists a pair (Π, σ) and a biholomorphism $X \xrightarrow{\simeq} X(\Pi, \sigma)$ such that C_0 corresponds to the image via $q_{\Pi,\sigma}$ of the first exceptional curve $\Pi_0^{-1}(O_{-1})$. We will identify X with $X(\Pi, \sigma)$ via this biholomorphism.

For any $i \in \{0, \ldots, n-2\}$ consider a small closed ball $\bar{B}_i$ centered at $O_i \in B^{\Pi_0 \circ \cdots \circ \Pi_i}$ and denote by S_i the lift of its boundary to B^Π. It is also convenient to denote by $S_{-1} := \partial\bar{B}^\Pi$, $\bar{B}_{n-1} := \sigma(\bar{B})$, and $S_{n-1} := \partial\bar{B}_{n-1} = \sigma(\partial\bar{B})$. In this way A decomposes as the union

$$A = \bigcup_{i=0}^{n-1} \mathfrak{A}_i \ ,$$

where each $\mathfrak{A}_i$ is a compact surface with two boundary components: a strictly pseudo-convex component $\partial_-(\mathfrak{A}_i) \simeq S_{i-1}$ and the strictly pseudo-concave component $\partial_+(\mathfrak{A}_i) \simeq S_i$. Every $\mathfrak{A}_i$ is biholomorphic to the manifold obtained from a blown up closed ball by removing a small open ball centered at a point of the exceptional divisor.

The universal covering space $\widetilde{X}$ can be obtained as an infinite union of copies A_k of A, the pseudo-convex boundary component of A_k being identified with the pseudo-concave boundary component of the previous annulus A_{k-1} via $\sigma \circ \Pi$. We will identify A_0 with A, so A_0 decomposes as $\bigcup_{i=0}^{n-1} \mathfrak{A}_i$; correspondingly we get decompositions

$$\forall j \in \mathbb{Z} \ , \ A_j = \bigcup_{i=0}^{n-1} \mathfrak{A}_{nj+i} \ , \ \widetilde{X} = \bigcup_{s \in \mathbb{Z}} \mathfrak{A}_s \ ,$$

where $\mathfrak{A}_{nj+i}$ is a copy of $\mathfrak{A}_i$ for every $i \in \{0, \ldots, n-1\}$, and $\widetilde{X}$ is obtained by identifying the pseudo-convex boundary component $\partial_-(\mathfrak{A}_s)$ of any $\mathfrak{A}_s$ with the pseudo-concave boundary component $\partial_+(\mathfrak{A}_{s-1})$ of the previous piece, in the obvious way. For every

$k \in \mathbb{Z}$ we put

$$\widetilde{X}_k := \bigcup_{s \leq k} \mathfrak{A}_s \ .$$

Note that $\widetilde{X}_{nj+n-1} = \bigcup_{j' \leq j} A_{j'}$. The boundary $\partial \widetilde{X}_k$ is the strictly pseudo-concave 3-sphere $\partial_+(\mathfrak{A}_k) \simeq \partial_+(\mathfrak{A}_{k-n\left[\frac{k}{n}\right]}) \simeq S_{k-n\left[\frac{k}{n}\right]}$. We denote by $\widehat{X}_k$ the surface obtained by gluing (a copy of) $\bar{B}_{k-n\left[\frac{k}{n}\right]}$ along this boundary. For every $k < l$ we have obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{X}_k & \stackrel{\iota_{k,l}}{\longrightarrow} & \widetilde{X}_l \\ \iota_k \downarrow & & \downarrow \iota_l \\ \widehat{X}_k & \stackrel{\pi_{k,l}}{\longleftarrow} & \widehat{X}_l \end{array} \ ,$$

where ι_k, ι_l, $\iota_{k,l}$ are the obvious inclusions, and $\pi_{k,l}$ is an order $l-k$ iterated blow up at the center of the ball $\bar{B}_{k-n\left[\frac{k}{n}\right]}$. For $k < l < m$ one has obviously $\iota_{l,m} \circ \iota_{k,l} = \iota_{k,m}$, $\pi_{k,l} \circ \pi_{l,m} = \pi_{k,m}$. Note that, for $i \in \{1, \ldots, n-1\}$ the blow up $\pi_{nj+i-1,nj+i}$ corresponds to Π_i via obvious identifications, whereas $\pi_{nj-1,nj}$ corresponds to $\Pi_0 \circ \sigma^{-1}$. We will denote by E_k the exceptional curve of the blow up $\pi_{k-1,k} : \widehat{X}_k \to \widehat{X}_{k-1}$. The pre-image $\iota_k^{-1}(E_k)$ is of course non-compact, but its Zariski closure in $\widetilde{X}$ is a compact rational curve $\widetilde{C}_k$ which can be explicitly obtained in the following way (see [4] for details):

Remark 4.2. *There exists $s \in \{1, \ldots, n+1\}$ such that the proper transform E_k^s of E_k in $\widehat{X}_{k+s}$ is contained in the image of ι_{k+s}, so $\widetilde{C}_k := \iota_{k+s}^{-1}(E_k^s)$ is a compact rational curve in $\widetilde{X}$, which can be identified with the Zariski closure of $\iota_k^{-1}(E_k)$.*

Note now that the group morphisms $H_2(\iota_k) : H_2(\widetilde{X}_k, \mathbb{Z}) \to H_2(\widehat{X}_k, \mathbb{Z})$ are isomorphisms, and that via the isomorphisms $H_2(\iota_k)$, $H_2(\iota_{k-1})$, the monomorphism $H_2(\iota_{k-1,k})$ defines a right splitting of the short exact sequence

$$0 \to \mathbb{Z}[E_k] \to H_2(\widehat{X}_k, \mathbb{Z}) \xrightarrow{H_2(\pi_{k-1,k})} H_2(\widehat{X}_{k-1}, \mathbb{Z}) \to 0 \ .$$

We denote by $\widetilde{e}_k$ the image in $H_2(\widetilde{X}_k, \mathbb{Z})$ of the class $[E_k]$ via $H_2(\iota_k)^{-1}$; to save on notations we will use the same symbol for the images of $\widetilde{e}_k$ in $H_2(\widetilde{X}_l, \mathbb{Z})$ $(l \geq k)$ and in $H_2(\widetilde{X}, \mathbb{Z})$. With these conventions we can write

$$H_2(\widetilde{X}_k, \mathbb{Z}) = \bigoplus_{s \leq k} \mathbb{Z}\widetilde{e}_s \ , \ H_2(\widetilde{X}, \mathbb{Z}) = \bigoplus_{s \in \mathbb{Z}} \mathbb{Z}\widetilde{e}_s$$

Using Remark 4.2 we see immediately that, via this decomposition, the homology class $[\widetilde{C}_k] \in H_2(\widetilde{X}, \mathbb{Z})$ decomposes as

$$[\widetilde{C}_k] = \widetilde{e}_k - \sum_{i=1}^{s_k} \widetilde{e}_{k+i} \text{ where } 1 \leq s_k \leq n+1 \ . \tag{7}$$

Note that one has obviously

$$s_k = -\widetilde{C}_k^2 - 1 \ . \tag{8}$$

The n compact curves of the minimal GSS surface $X = X(\Pi, \sigma)$ are just the projections $C_k := \alpha(\widetilde{C}_{nj+k})$ of the compact curves we obtained in the universal cover $\widetilde{X}$. Note also that the set $\{e_k := \alpha_*(\widetilde{e}_k) |\ k \in \{0, \ldots, n-1\}\}$ is precisely the standard basis $\mathfrak{B}$ of X.

We will show now that any class $\widetilde{e}_i$ decomposes formally in a well defined way as an infinite series of classes of compact curves which is bounded towards the pseudo-convex end. These formal identities correspond to equalities in the Borel-More homology of $\widetilde{X}$. We will begin with several interesting examples:

Exemple 4.3. *Enoki surfaces.*

An Enoki surface X has a cycle $\sum_{i=0}^{n-1} C_i$ of $n = b_2(X)$ rational curves. We have

$$[C_i] = e_i - e_{i+1}, \quad i \in \mathbb{Z}_n \ .$$

In the universal cover $\widetilde{X}$ we have curves $\widetilde{C}_i$ representing the 2-homology classes $[\widetilde{C}_i] = \widetilde{e}_i - \widetilde{e}_{i+1},\ i \in \mathbb{Z}$, therefore we get the decomposition

$$\widetilde{e}_i = \sum_{j=0}^{\infty} [\widetilde{C}_{i+j}] \ .$$

Exemple 4.4. *Minimal GSS surfaces X with $b_2(X) = 1$.*

There are two classes of minimal GSS surfaces X with $b_2(X) = 1$:

- Enoki surfaces with $b_2 = 1$,
- Inoue-Hirzebruch surfaces with one cycle,

The first class has been treated above, so we discuss the second one. The curves in the universal cover $\widetilde{X}$ of an Inoue-Hirzebruch surface X with $b_2(X) = 1$ form two chains of rational curves

$$\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i}, \quad \sum_{i\in\mathbb{Z}} \widetilde{C}_{2i+1}$$

with $[\widetilde{C}_i] = \widetilde{e}_i - (\widetilde{e}_{i+1} + \widetilde{e}_{i+2})$. Using the Fibonacci sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$u_0 = 1, u_1 = 1,\ u_n = u_{n-2} + u_{n-1}, n \geq 2$$

we get

$$\widetilde{e}_i = \sum_{n\geq 0} u_n \widetilde{C}_{i+n} \ .$$

Exemple 4.5. *Minimal GSS surfaces X with $b_2(X) = 2$.*

There are four classes of minimal GSS surfaces X with $b_2(X) = 2$.

- Enoki surfaces with $b_2 = 2$,
- Intermediate surfaces,
- Inoue-Hirzebruch surfaces with one cycle,
- Inoue-Hirzebruch surface with two cycles consisting of a rational curve with a double point.

The case of Enoki surfaces has been treated above for any $b_2 > 0$.

Intermediate surfaces. An intermediate surface with $b_2 = 2$ has a rational curve with a double point C_0 and a non-singular rational curve C_1 with

$$C_0^2 = -1, \quad C_1^2 = -2, \quad C_0 C_1 = 1 \ .$$

Using our conventions and notations we get decompositions

$$[C_0] = -e_1,\ [C_1] = e_1 - e_0 \ .$$

In the universal covering space $\widetilde{X}$ we have an infinite chain of curves $\sum_{i\in\mathbb{Z}} \widetilde{C}_{2i}$ with pairwise disjoint trees $\widetilde{C}_{2i+1}$, $i \in \mathbb{Z}$ which decompose as follows:

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}\ ,\ [\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} \ .$$

We have then

$$\begin{aligned} \widetilde{e}_{2i} &= \sum_{j=0}^{\infty} 2^j \Big\{ (\widetilde{e}_{2(i+j)} - \widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2}) + (\widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2}) \Big\} \\ &= \sum_{j=0}^{\infty} 2^j \Big\{ \widetilde{C}_{2(i+j)} + \widetilde{C}_{2(i+j)+1} \Big\} \end{aligned}$$

$$\widetilde{e}_{2i+1} = (\widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}) + \widetilde{e}_{2i+2} = [\widetilde{C}_{2i+1}] + \sum_{j=0}^{\infty} 2^j \Big\{ [\widetilde{C}_{2(i+j+1)}] + [\widetilde{C}_{2(i+j+1)+1}] \Big\} \ .$$

Inoue-Hirzebruch surfaces with $b_2 = 2$ and one cycle. Such a surface has a cycle $C_0 + C_1$ of non-singular rational curves, where

$$C_0^2 = -4 \ , \ C_1^2 = -2 \ , \ C_0 C_1 = 2 \ .$$

More precisely

$$[C_0] = -2e_1 \ , \ [C_1] = e_1 - e_0 \ .$$

In the universal cover $\widetilde{X}$ the homology classes of the compact curves $\widetilde{C}_i$ decompose as

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} - \widetilde{e}_{2i+3} \ , \ [\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} \ .$$

These curves form two disjoint infinite chains of rational curves

$$\sum_{i \in \mathbb{Z}} \widetilde{C}_{4i} + \widetilde{C}_{4i+3} \ , \ \sum_{i \in \mathbb{Z}} \widetilde{C}_{4i+1} + \widetilde{C}_{4i+2} \ .$$

We define by induction the following sequences of positive integers $(a_j)_{j \in \mathbb{N}}$, $(b_j)_{j \in \mathbb{N}}$ where $a_0 = b_0 = 1$, and

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = A^j \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \text{ with } A := \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \ .$$

Then

$$\begin{aligned} \widetilde{e}_{2i} &= \sum_{j=0}^{\infty} \Big\{ a_j \big(\widetilde{e}_{2(i+j)} - \widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2} - \widetilde{e}_{2(i+j)+3} \big) + b_j \big(\widetilde{e}_{2(i+j)+1} - \widetilde{e}_{2(i+j)+2} \big) \Big\} \\ &= \sum_{j=0}^{\infty} a_j [\widetilde{C}_{2(i+j)}] + b_j [\widetilde{C}_{2(i+j)+1}] \\ &= [\widetilde{C}_{2i}] + [\widetilde{C}_{2i+1}] + 2[\widetilde{C}_{2i+2}] + 3[\widetilde{C}_{2i+3}] + 5[\widetilde{C}_{2i+4}] + 7[\widetilde{C}_{2i+5}] + 12[\widetilde{C}_{2i+6}] + \cdots \end{aligned}$$

$$\widetilde{e}_{2i+1} = (\widetilde{e}_{2i+1} - \widetilde{e}_{2i+2}) + \widetilde{e}_{2i+2} = [\widetilde{C}_{2i+1}] + \sum_{j=0}^{\infty} a_j [\widetilde{C}_{2(i+j+1)}] + b_j [\widetilde{C}_{2(i+j+1)+1}] \ .$$

Inoue-Hirzebruch surfaces with $b_2 = 2$ and two cycles. In this case we have two cycles C_0, C_1 consisting of a rational curve with double point. The intersection numbers are

$$C_0^2 = -1, \quad C_1^2 = -1 \ ,$$

and the decompositions of $[C_i]$ with respect to the standard basis are

$$[C_0] = -e_1, \quad [C_1] = -e_0 \ .$$

In $\widetilde{X}$ we have two disjoint chains of rational curves $\sum_{i \in \mathbb{Z}} \widetilde{C}_{2i}$, $\sum_{i \in \mathbb{Z}} \widetilde{C}_{2i+1}$, where

$$[\widetilde{C}_{2i}] = \widetilde{e}_{2i} - \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} \ , [\widetilde{C}_{2i+1}] = \widetilde{e}_{2i+1} - \widetilde{e}_{2i+2} - \widetilde{e}_{2i+3} \ .$$

Therefore denoting again by $(u_j)_{j \in \mathbb{N}}$ the Fibonacci sequence used before we obtain:

$$\begin{aligned} \widetilde{e}_{2i} &= \sum_{j \geq 0} u_j (\widetilde{e}_{2i+j} - \widetilde{e}_{2i+j+1} - \widetilde{e}_{2i+j+2}) = \sum_{j \geq 0} u_j [\widetilde{C}_{2i+j}] \\ \widetilde{e}_{2i+1} &= \sum_{j \geq 0} u_j (\widetilde{e}_{2i+j+1} - \widetilde{e}_{2i+j+2} - \widetilde{e}_{2i+j+3}) = \sum_{j \geq 0} u_j [\widetilde{C}_{2i+j+1}] \ . \end{aligned}$$

The next result deals with the general case:

Proposition 4.6. *Let $X = X(\Pi, \sigma)$ be a GSS surface with $b_2(X) = n \geq 1$ and $(\widetilde{e}_i)_{i\in\mathbb{Z}}$ be the standard base of $H_2(\widetilde{X}, \mathbb{Z})$ constructed above. Then for any $i \in \mathbb{Z}$ there exists a well defined infinite series with positive integer coefficients $\sum_{j\geq 0} \alpha^i_j[\widetilde{C}_{i+j}]$ whose sum is the image of $\widetilde{e}_i$ in the Borel-Moore homology group $H_2^{\mathrm{BM}}(\widetilde{X}, \mathbb{Z})$. Moreover, the sequence $(\alpha^i_j)_{j\geq 0}$ is always increasing, has $\alpha^i_0 = 1$, and*

$$\begin{cases} \forall j, \quad \alpha^i_j = 1 & \text{iff} \quad X \text{ is a Enoki surface} , \\ \\ \lim_{j\to\infty} \alpha^i_j = \infty & \text{iff} \quad X \text{ is not a Enoki surface} . \end{cases}$$

Proof. We determine the coefficient α^i_k of $[\widetilde{C}_{i+k}]$ by induction on $k \geq 0$ such that we have the congruence

$$\widetilde{e}_i \equiv \sum_{j=0}^{k} \alpha^i_j[\widetilde{C}_{i+j}] \text{ modulo classes } \widetilde{e}_j \ , \ j > k \ . \tag{9}$$

For $k = 0$ we use formula (7) to obtain

$$[C_i] = \widetilde{e}_i - \sum_{j=1}^{s_i} \widetilde{e}_{i+j} \equiv \widetilde{e}_i \text{ mod classes } \widetilde{e}_{i+j} \ , \ j > 0 \ ,$$

so we get a well-defined solution $\alpha^i_0 = 1$.

Suppose now that the development is determined till $k \in \mathbb{N}$ so that

$$\widetilde{e}_i \equiv \sum_{j=0}^{k} \alpha^i_j[\widetilde{C}_{i+j}] = \sum_{j=0}^{k} \alpha^i_j\big(\widetilde{e}_{i+j} - \sum_{l=1}^{s_{i+j}} \widetilde{e}_{i+j+l}\big) \text{ mod classes } \widetilde{e}_{i+j} \ , \ j > k \ .$$

We define $J^i_{k+1} = \big\{j \in \{0, \ldots, k\} \mid \ i + j + 1 \leq i + k + 1 \leq i + j + s_{i+j}\big\}$. With this notation we obtain a well defined solution

$$\alpha^i_{k+1} = \sum_{l\in J^i_{k+1}} \alpha^i_l \tag{10}$$

which guarantees the congruence (9) for $k + 1$.

We prove now that the obtained sequence $(\alpha^i_j)_{j\in\mathbb{N}}$ is increasing as claimed. Since for any $l \in \mathbb{Z}$ it holds $l + 1 \leq l + s_l$, we have $k + 1 \in J^i_k$. Therefore α^i_k intervenes in the decomposition (10), so $\alpha^i_{k+1} \geq \alpha^i_k$.

If X is an Enoki surface, then $\alpha_{i+j} = 1$ for every $j \geq 0$ as we have seen in Example 4.3. If X is not an Enoki surface there is at least one curve $\widetilde{C}_{i+j}$ such that $\widetilde{C}^2_{i+j} \leq -3$, hence, using again formula (7) we see that $\widetilde{e}_{i+j+2}$ appears in the decompositions of the curves $\widetilde{C}_{i+j}$ and $\widetilde{C}_{i+j+1}$, hence $j, \ j + 1 \in J^i_{j+2}$ and $\alpha_{j+2} \geq \alpha_j + \alpha_{j+1} > \alpha_{j+1}$. Since the configuration of the curves in $\widetilde{X}$ is periodic, we have $\widetilde{C}^2_{i+j+kn} = \widetilde{C}^2_{i+j} \leq -3$ for every $k \in \mathbb{N}$, so $\alpha_{j+kn+2} > \alpha_{j+kn+1}$, which proves the result. □

Remark 4.7. *The decomposition $\widetilde{e}_i = \sum_{j\geq 0} \alpha^i_j[\widetilde{C}_{i+j}]$ given by Proposition 4.6 is the only decomposition of the image of $\widetilde{e}_i$ in $H_2^{\mathrm{BM}}(\widetilde{X}, \mathbb{Z})$ as the sum of a series – bounded towards the pseudo-convex end – of classes of compact curves.*

Proof. Indeed, it suffices to see that if a sum $\sum_{i\geq k} a_i[\widetilde{C}_i]$ vanishes in $H_2^{\mathrm{BM}}(X, \mathbb{Z})$, then all coefficients a_i vanish. This follows easily using the Poincaré duality isomorphism $PD : H_2^{\mathrm{BM}}(\widetilde{X}, \mathbb{Z}) \xrightarrow{\simeq} H^2(\widetilde{X}, \mathbb{Z})$. Using the geometric interpretation of the Poincaré duality in terms of intersection numbers we get

$$0 = \langle PD(\sum_{i\geq k} a_i[\widetilde{C}_i]), \widetilde{e}_k\rangle = -a_k \ .$$

By induction we get $a_i = 0$ for all $i \geq k$. □

Note that for an Enoki surface X the classes $[\widetilde{C}_i]$ are linearly independent in $H_2(\widetilde{X},\mathbb{Z})$, but $\sum_{i\in\mathbb{Z}}[\widetilde{C}_i]=0$ in $H_2^{\mathrm{BM}}(\widetilde{X},\mathbb{Z})$.

Let now $p:\mathcal{X}\to B$ be a holomorphic family of GSS surfaces and let

$$e=(e_b)_{b\in B}\in H^0(B,\underline{H}_2)\ ,\ \widetilde{e}=(\widetilde{e}_b)_{b\in B}\in H^0(B,\underline{\widetilde{H}}_2)$$

be families of homology classes as in the first extension theorem Theorem 3.1.

Our goal now is to identify explicitly the effective divisor $\widetilde{E}_{b_0}$ for $b_0\in H_e$. As we explain in the introduction this divisor is interesting because it can be written as a limit of simply exceptional divisors $\widetilde{E}_{b_n}$ for a sequence $b_n\to b_0$, $b_n\in B_e$.

We suppose for simplicity that X_{b_0} is minimal because, if not, we can blow down the exceptional divisors in X_{b_0} (and their deformations in X_b for b around b_0) and reduce the problem to the case of a minimal central fiber.

Theorem 4.8. *Under the assumptions and with the notations above, the effective divisor $\widetilde{E}_{b_0}$ is the sum of the infinite series $\sum_{j\geq 0}\alpha_j^i\widetilde{C}_{i+j}$ given by Proposition 4.6.*

Proof. Our extension theorem yields a divisor $\widetilde{\mathcal{E}}\subset\widetilde{\mathcal{X}}$ flat over B whose fiber over b_0 is $\widetilde{E}_{b_0}$, so the class defined by $\widetilde{E}_b$ in Borel-Moore homology is $\widetilde{e}_b$ for any $b\in B$. The result follows directly from Remark 4.7 and Lemma 4.9 below. □

Lemma 4.9. *Let X be a GSS surface with $b_2(X)>0$ and $\alpha:\widetilde{X}\to X$ its universal cover. Then $\widetilde{X}$ does not contain any irreducible non-compact 1-dimensional analytic subset bounded towards the pseudo-convex end.*

Proof. Suppose that there exists such a subspace $S\subset\widetilde{X}$. We can suppose that X is minimal and was identified with $X(\Pi,\sigma)$ as explained above. Since S is bounded towards the pseudo-convex end we can find k sufficiently large such that S does not intersect the closure of $\widetilde{X}_k$.

We have a natural surjective *holomorphic* map $q_k:\widetilde{X}\to\widehat{X}_k$ which contracts all compact curves $\widetilde{C}_l$ for $l>k$ to a point $\widehat{O}_k\in\widehat{X}_k$ (the center of ball $B_{k-n\left[\frac{k}{n}\right]}$ involved in the construction of $\widehat{X}_k$) and defines a biholomorphism

$$\widetilde{X}\setminus\bigcup_{l>k}\widetilde{C}_l\xrightarrow{\simeq}\hat{X}_k\setminus\{\widehat{O}_k\}$$

(see [4]). Since S does not intersect the closure of $\widetilde{X}_k$ the image $S_0:=q_k(S)$ is contained in the complement of this closure in $\hat{X}_k$, which is the open ball $B_{k-n\left[\frac{k}{n}\right]}$.

But then $S_0\setminus\{\widehat{O}_k\}$ is a closed 1-dimensional subspace of a punctured open 2-ball, so its closure in this ball will be a closed 1-dimensional subspace of the ball, by the second Remmert-Stein theorem. This extension would be compact, which is of course impossible.

□

Coming back to our second extension theorem Theorem 3.5, and taking into account Theorem 4.8 we see that, when a fiber X_b ($b\in A$) of the family is a GSS surface, the obtained effective divisors $\widetilde{E}_b$ representing a lift $\widetilde{e}_b$ are always given by series of *compact* curves. We end our article with a theorem (mentioned in the abstract and the introduction) which gives an interesting motivation of our results: conversely, if in the conditions of Theorem 3.5, for a point $b\in A$ all the irreducible components of the divisors $\widetilde{E}_b$ are compact, then X_b has b_2 rational curves, so it is a GSS surface. We believe that *the irreducible components of the divisors $\widetilde{E}_b$ are always compact*, which would have a very important consequence: the main conjecture **C** holds for any class VII surface which fits as the central fiber of a bidimensional family $\mathcal{X}\to\Delta\subset\mathbb{C}^2$ whose fibers X_z, $z\neq 0$ are GSS surfaces. Therefore, for proving conjecture **C** for this class

of surfaces, it suffices to prove that for a sequence $(b_n)_n$ of $\Delta \setminus \{0\}$ converging to 0, open irreducible components cannot appear in the limit process $\widetilde{E}_{b_n} \to \widetilde{E}_0$, but only *infinite bubbling* i.e. the appearance of infinitely many compact irreducible components.

Let X be a class VII surface with $b_2 > 0$ having the oriented topological type of $(S^1 \times S^3)\#n\overline{\mathbb{P}}^2$. In the same way as explained at the beginning section for GSS surfaces we obtain a unique (unordered) basis $\mathfrak{B} \subset H_2(X, \mathbb{Z})$ trivializing the intersection form q_X of X such that $c_1(\mathcal{K}_X) = \sum_{e \in \mathfrak{B}} PD(e)$. This basis will be called again *the standard basis* of $H_2(X, \mathbb{Z})$. Note that the homology class of any simply exceptional divisor of X (if such a divisor exists) is an element of this standard basis. Any compact curve $C \subset \widetilde{X}$ is a smooth rational curve. Indeed, its projection $\alpha(C)$ on X is either a

(1) smooth rational curve, or
(2) a rational curve with a simple singularity, or
(3) or an elliptic curve

(see [9] p. 399). But in the last two cases $\alpha(C)$ would be a cycle in X, and it is known that the image of its fundamental group in $\pi_1(X)$ of a cycle is infinite (see [9] p. 404). In the second case $\alpha^{-1}(\alpha(C))$ is a countable union of smooth rational curves, and the third case is excluded, because the pre-image of such a cycle does not contain any compact curve.

Therefore

$$0 = g_a(C) = 1 + \frac{1}{2}\langle c_1(\mathcal{K}_{\tilde{X}}(C)), C\rangle \tag{11}$$

Since all the elements of the standard basis $\mathfrak{B}$ are represented by 2-spheres, they can be lifted to $\widetilde{X}$, so we get a basis $\widetilde{\mathfrak{B}}$ of $H_2(\widetilde{X}, \mathbb{Z})$ which is mapped on $\mathfrak{B}$. We can identify naturally $H^2(\widetilde{X}, \mathbb{Z})$ with $\prod_{\widetilde{e} \in \widetilde{\mathfrak{B}}} \mathbb{Z} PD(\widetilde{e})$, and we obtain

$$c_1(\mathcal{K}_{\tilde{X}}) = \alpha^*(c_1(\mathcal{K}_X)) = \alpha^*(\sum_{e \in \mathfrak{B}} PD(e)) = \sum_{\widetilde{e} \in \widetilde{\mathfrak{B}}} PD(\widetilde{e}) \ .$$

Decomposing $[C] = \sum_{\widetilde{e} \in \widetilde{\mathfrak{B}}} x_{\widetilde{e}} \widetilde{e}$ (with finitely many non-zero coefficients $x_{\widetilde{e}}$) and using the identities $\langle PD(\widetilde{e}), \widetilde{f}\rangle = -\delta_{\widetilde{e}\widetilde{f}}$ equation (11) becomes:

$$\sum_{\widetilde{e} \in \widetilde{\mathfrak{B}}} x_{\widetilde{e}}(1 + x_{\widetilde{e}}) = 2 \ ,$$

which can hold only if there exists $\widetilde{e}_C \in \widetilde{\mathfrak{B}}$ and a finite set $\widetilde{\mathfrak{B}}_C \subset \widetilde{\mathfrak{B}}$ with $\widetilde{e}_C \notin \widetilde{\mathfrak{B}}_C$ such that either:

a) $[C] = \widetilde{e}_C - \sum_{\widetilde{e} \in \widetilde{\mathfrak{B}}_C} \widetilde{e}$, or
b) $[C] = -2\widetilde{e}_C - \sum_{\widetilde{e} \in \widetilde{\mathfrak{B}}_C} \widetilde{e}$.

We say that a curve $C \subset \widetilde{X}$ is of type (a) or (b) if the decomposition of $[C]$ has the form a) or b) respectively.

Theorem 4.10. *Let X be a minimal class VII surface with $b_2(X) > 0$ having the topological type $(S^1 \times S^3)\#n\overline{\mathbb{P}}^2$. Suppose that for every $e \in \mathfrak{B}$ there exists a lift $\widetilde{e} \in \widetilde{\mathfrak{B}}$ and an effective divisor $\widetilde{E} \subset \widetilde{X}$ representing the image of $\widetilde{e}$ in $H_2^{\mathrm{BM}}(X, \mathbb{Z})$ and whose irreducible components are all compact. Then X has $b_2(X)$ rational curves, so it is a GSS surface.*

Proof. The Borel-Moore homology group $H_2^{\mathrm{BM}}(\widetilde{X}, \mathbb{Z})$ can be identified with $\prod_{\widetilde{e} \in \widetilde{\mathfrak{B}}} \mathbb{Z}\widetilde{e}$. Since $\widetilde{E}$ represents the class $\widetilde{e}$, it follows that there exists an irreducible component C_e of $\widetilde{E}$ which has type (a) with $\widetilde{e}_{C_e} = e$. The image $D_e := \alpha(C_e)$ is either a smooth rational curve, or a singular rational curve with a double point. In both cases the

irreducible components of $\alpha^{-1}(D_e)$ are $f^n(C_e)$ $(n \in \mathbb{Z})$, where f is a generator of the automorphism group $\mathrm{Aut}_X \widetilde{X} \simeq \mathbb{Z}$.

For $e \neq e' \in \mathfrak{B}$ we see that the homology classes $[C_e]$, $[C_{e'}]$ are not congruent modulo $\mathrm{Aut}_X \widetilde{X}$, because the first terms in the decompositions of these classes are the lifts $\widetilde{e}$, $\widetilde{e'}$. Therefore $D_e \neq D_{e'}$. In this way we obtain $n = b_2(X)$ rational curves in X, proving that X is a GSS surface, by the main result in [6]. □

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